AMERICAN NUCLEAR SOCIETY

555 North Kensington Avenue
La Grange Park, Illinois
60526-5592 USA

Tel: 708 / 352-6611
E-Mail: NUCLEUS@ans.org
http: //www.ans.org
Fax: 708 / 352-0499

AUTHOR PROOF INSTRUCTIONS

Your paper is scheduled for publication in a forthcoming issue of *Nuclear Science and Engineering*. You are receiving your author proofs electronically. You will not receive anything in the regular mail. Please follow these procedures:

1. Download your author proof, print it out, and review it.

2. Proofread your paper very carefully. This will be your final reading before publication. This material has been copyedited and typeset from the disk/electronic file that you provided with your manuscript or from the hard copy that you provided, and it has been transmitted to you through the Internet. This process produces excellent results, but please check the following especially carefully: translation of Greek letters, accented letters, special characters, superscripts and subscripts, and mathematical symbols; typesetting of mathematics; and formatting of tables. Also, note that your manuscript has been copyedited according to ANS technical journal style; please check that technical meanings have been unaffected by the copyediting. Please limit your corrections to those that are absolutely necessary because changes at this stage of processing are both time-consuming and expensive.

3. Please note that the typesetter has used low-resolution electronically scanned versions of your figures as FPOs (for position only). These will be replaced with high-resolution versions at the printing stage.

4. Necessary corrections should be sent as follows: (a) a list of corrections sent by electronic mail and/or (b) your manuscript with your corrections marked on it sent by fax (your fax should include a list of your corrections). If you have been sent a list of queries, we would greatly appreciate your reply to these queries either by electronic mail or by fax.

5. Please return your corrections within a week of receipt of your author proofs. Note that delay in transmitting your corrections may delay publication of your paper or may require publication without your corrections.

6. Return proofs using the following name and electronic mail address or fax number:

Suzanne Pals
E-mail: spals@ans.org
Fax: 708-352-6464

Thank you for your prompt attention to these proofs. If you have any questions, please feel free to contact me.

Suzanne Pals
Staff Editor

QUERIES

1. Is Eq. (4) correct?
2. Is pdf okay here?
3. Please insert section numbers.
4. Is “cumulative density function” correct?
5. Please insert equation numbers.
6. Is “monotonically” (or “monotonic”) correct?
7. Is the inserted section number correct?
8. There seems to be a missing word in this sentence. Please correct.

NUCLEAR SCIENCE AND ENGINEERING: **150**, 1–25 (2005)

Modeling PSA Problems—I: The Stimulus-Driven Theory of Probabilistic Dynamics

P. E. Labeau*

Université Libre de Bruxelles (CP165/84)
Av. F.D. Roosevelt 50, B - 1050 Brussels, Belgium

and

J. M. Izquierdo

Consejo de Seguridad Nuclear
Justo Dorado 11, E - 28040 Madrid, Spain

Received January 22, 2003
Accepted September 7, 2004

Abstract—*The theory of probabilistic dynamics (TPD) offers a framework capable of modeling the interaction between the physical evolution of a system in transient conditions and the succession of branchings defining a sequence of events. Nonetheless, the Chapman-Kolmogorov equation, besides being inherently Markovian, assumes instantaneous changes in the system dynamics when a setpoint is crossed. In actuality, a transition between two dynamic evolution regimes of the system is a two-phase process. First, conditions corresponding to the triggering of a transition have to be met; this phase will be referred to as the activation of a "stimulus." Then, a time delay must elapse before the actual occurrence of the event causing the transition to take place. When this delay cannot be neglected and is a random quantity, the general TPD can no longer be used as such. Moreover, these delays are likely to influence the ordering of events in an accident sequence with competing situations, and the process of delineating sequences in the probabilistic safety analysis of a plant might therefore be affected in turn. This paper aims at presenting several extensions of the classical TPD, in which additional modeling capabilities are progressively introduced. A companion paper sketches a discretized approach of these problems.*

I. INTRODUCTION

As an accident transient develops after the occurrence of an initiating event perturbing the steady-state working conditions of a plant, the description of its dynamic evolution has to be supplemented by giving all possible causes of possibly stochastic bifurcation between deterministic sections of a trajectory in the space of process variables. These changes in the system dynamics are due either to stochastic hardware failures or to automatic control-protection or operator-driven actions aiming at mitigating the accident. In turn, the value taken by the process variables can significantly affect the probability of a transition between two dynamic evolution modes. This statement is obvious when the dynamics is modified by the activation of a protection device after the crossing of a threshold (i.e., setpoint) on the process variables. But, the failure rate of hardware components is also likely to be influenced by variations in temperature or pressure, for instance.

This close interaction between the process variables evolution and the succession of events defining an accident scenario is at the heart of the modeling of accident propagation in industrial systems, such as nuclear power plants. Yet, it has not received sufficient consideration, or at least sufficient visibility, in the event tree/fault tree methodology typically used in conventional probabilistic safety analysis (PSA) studies.[1,2]

*E-mail: pelabeau@ulb.ac.be

Observing and formalizing this interaction process gave rise to the development of the theory of probabilistic dynamics (TPD), also known as the theory of continuous event trees.[3] The original paper accounted only for stochastic transitions between system states, but the theory was generalized to setpoint transitions,[4] and it then fully appeared as an extension of classical event trees. Probabilistic dynamics is also known to put within a common framework different previous attempts at bringing the dynamics into the sequence delineation problem.[5]

Up to now and mainly for setpoint transitions, efforts to reinterpret the actual engineering practice in terms of the TPD equations led to confirmation of the approach.[6] However, the studies that were performed[7] indicated

- the difficulties associated with ensuring the overall consistency of a correct sequence delineation
- the need to extend the TPD in order to incorporate house event information as well as operator delays.

Other theoretical as well as practical applications, mostly in the context of level-1 PSA studies, are reported in Ref. 8. A nonnuclear case, pertaining to aeronautics, is described in Ref. 9. More recently, a Monte Carlo–based level-2 application of the TPD was propounded in Ref. 10. Also, when considering level-2 problems, or in general continuous event trees in which branchings have a high uncertainty in the occurrence of phenomena (and are therefore more stochastic), the TPD needs some adaptations in order to obtain an equivalent consistency. These extensions are the subject of this work. A companion paper[11] sketches a discretized treatment of these new problems.

The main concept introduced in this extension of the theory is that of stimulus activation, which must take place prior to the actual transition between two system configurations corresponding to different dynamic evolutions. A stimulus, which is usually defined in correspondence to some specific values of the process variables, can for instance be a signal initiating the decision-making process of the operator team or the crossing of a setpoint triggering the action of an automatic protection device. It could also correspond to the entry of the system in a region of phase-space where ignition criteria are fulfilled. In general, the term "stimulus" covers any situation that potentially causes, after a given time delay, an event to occur and subsequently a branching to take place in the continuous event tree.

Time delays are of paramount importance in this description of an accident progression. Indeed, the competition between events determining the sequence delineation is driven as before by the minimum time to the occurrence of an event, but this process is now given by the sum of two (possibly) random times: the time interval necessary to reach a zone in phase-space where a stimulus is activated and the time delay following this activation before the system dynamics is actually modified. This more complex interaction has to be modeled in the dynamic reliability framework.

This paper is organized so as to highlight the evolution of the methodology. In order to do so, the fundamental aspects of the TPD are reviewed and summarized in Sec. II, before its extension to a semi-Markov treatment is presented in a slightly reformulated fashion and discussed. Section III displays original adaptations of the semi-Markov TPD in order to account for specific aspects of PSA. It first deals with instantaneous and random variations of the process variables; then, it introduces the concept of stimulus and how it can be implemented within the semi-Markov theory. This latter assumption is—partly—released in Sec. IV, where a non-Markov treatment is provided. It is then proven that this latter modeling easily reduces to the setpoint approach of Ref. 4 if appropriate simplifications are brought into the equations. A test case is presented in Sec. V, illustrating the new concepts and showing the coherence of the theory. Concluding remarks are then provided.

II. THE THEORY OF CONTINUOUS EVENT TREES

The Markovian version of the TPD has been extensively described in Refs. 3 and 12, both in differential and integral forms. We summarize the main aspects of the latter case in Sec. II.A and explain how this theory relates to a branching process, allowing a dynamic approach to PSA. As the dynamic aspects of man-machine interactions in accident transients were incorporated, models of the human operator were soon envisioned for inclusion in dynamic PSA (Refs. 13 and 14). Yet, these aspects require a broader framework than the purely Markovian one, and this initial extension of the theory is reproduced as a starting point for new semi-Markovian and non-Markovian extensions entailed by the level-2 PSA constraints.

II.A. Integral Equations of the Markovian TPD

Let $\bar{x}$ be the vector of process variables describing the dynamic behavior of the plant. We denote by i the group of system configurations in which the dynamic evolution is given by the equivalent explicit form

$$\bar{x}(t) = \bar{g}_i(t, \bar{x}_o) \ , \quad \bar{x}_o = \bar{g}_i(o, \bar{x}_o) \ . \tag{1}$$

The branchings in an accident sequence correspond to transitions between two dynamics j and i, which are characterized by transition rates $p(j \to i|\bar{x})$, possibly dependent on the process variables value but explicitly independent of time in the Markovian case. The total transition rate out of configuration j is written as

$$\lambda_j(\bar{x}) = \sum_{i \neq j} p(j \to i|\bar{x}) \quad . \tag{2}$$

The integral form of the Chapman-Kolmogorov (C.K.) equation gives the evolution of the probability density function (pdf) $\pi(\bar{x},i,t)$ of finding the plant in a configuration i with process variables $\bar{x}$ a time t after the beginning of the transient. We can express the result given in Ref. 3 in terms of the outgoing density $\psi(\bar{x},i,t)$ leaving configuration i at $\bar{x},t$:

$$\psi(\bar{x},i,t) \equiv \lambda_i(\bar{x})\pi(\bar{x},i,t) \quad , \tag{3}$$

whose evolution is given by

$$\begin{aligned}\psi(\bar{x},i,t) = & \int \pi(\bar{u},i,o)\delta(\bar{x}-\bar{g}_i(t,\bar{u})) \\ & \times \lambda_i(\bar{x})e^{-\int_o^t \lambda_i(\bar{g}_i(s,\bar{u}))\,ds}\,d\bar{u} \\ & + \sum_{j\neq i}\int_o^t d\tau \int d\bar{u}\,\psi(\bar{u},j,\tau)\hat{p}(j\to i|\bar{u}) \\ & \times \delta(\bar{x}-\bar{g}_i(t-\tau,\bar{u})) \\ & \times \lambda_i(\bar{x})e^{-\int_o^{t-\tau}\lambda_i(\bar{g}_i(s,\bar{u}))\,ds} \quad ,\end{aligned} \tag{4}$$

where $\hat{p}(j\to i|\bar{u}) \equiv p(j\to i|\bar{u})/\lambda_j(\bar{u})$ is the probability of a transition to configuration i, given configuration j is exited at point $\bar{u}$ in the process variables space.

1 The interpretation of Eq. (4) is direct: The plant leaves configuration i at time t with process variables $\bar{x}$ either if it has been in configuration i from the beginning of the transient, following dynamics $\bar{g}_i(t,\bar{u})$ during a time distributed according to $\lambda_i(\bar{x})\exp(-\int_o^t \lambda_i(\bar{g}_i(s,\bar{u}))\,ds)$, or if the last transition to configuration i took place at time $\tau < t$, where the system left configuration j and the sojourn time $t-\tau$ in dynamics i was again given by the exponential distribution.

How can Eq. (4) bring insight into the sequence generation process? The link between the mathematical expressions and the branching process comes from the formal development of $\psi(\bar{x},i,t)$ in Neumann series (see Ref. 15):

$$\psi(\bar{x},i,t) = \sum_{n=o}^{\infty}\psi^{(n)}(\bar{x},i,t) \quad . \tag{5}$$

It can be easily checked[4] from Eqs. (3), (4), and (5) that

$$\begin{aligned}\psi^{(o)}(\bar{x},i,t) = & \int \pi(\bar{u},i,o)\lambda_i(\bar{x})\delta(\bar{x}-\bar{g}_i(t,\bar{u})) \\ & \times e^{-\int_o^t \lambda_i(\bar{g}_i(s,\bar{u}))\,ds}\,d\bar{u}\end{aligned} \tag{6a}$$

and

$$\begin{aligned}\psi^{(n)}(\bar{x},i,t) = & \sum_{j\neq i}\int_o^t d\tau\int d\bar{u}\,\hat{p}(j\to i|\bar{u})\psi^{(n-1)}(\bar{u},j,\tau) \\ & \times \delta(\bar{x}-\bar{g}_i(t-\tau,\bar{u})) \\ & \times \lambda_i(\bar{x})e^{-\int_o^{t-\tau}\lambda_i(\bar{g}_i(s,\bar{u}))\,ds} \quad n \geq 1 \quad ;\end{aligned} \tag{6b}$$

$\psi^{(n)}(\bar{x},i,t)$ is directly understood as the outgoing density of i at point $\bar{x}$ and time t after n previous transitions between system configurations. In an event tree interpretation of the process, $\psi^{(n)}(\bar{x},i,t)$ appears as the density of branching out of dynamics i at $\bar{x}$, t, given n branchings had already taken place after the occurrence of the initiating event. These densities in the different system dynamics can be iteratively calculated from Eq. (6b), and the probability density to be in a given dynamics (i.e., on a given branch) after n branchings is written

$$\begin{aligned}\pi^{(n)}(\bar{x},i,t) = & \sum_{j\neq i}\int_o^t d\tau\int d\bar{u}\,\psi^{(n-1)}(\bar{u},j,\tau)\hat{p}(j\to i|\bar{u}) \\ & \times \delta(\bar{x}-\bar{g}_i(t-\tau,\bar{u})) \\ & \times e^{-\int_o^{t-\tau}\lambda_i(\bar{g}_i(s,\bar{u}))\,ds} \quad .\end{aligned} \tag{7}$$

It was shown in Ref. 4 that classical event trees can be rigorously derived from this theory when considering only setpoint transitions, i.e., transitions taking place when thresholds on the process variables are crossed.

An alternative presentation is based on the ingoing density $\varphi(\bar{x},i,t)$ into dynamics i, which is defined by

$$\begin{aligned}\varphi(\bar{x},i,t) & = \sum_{j\neq i} p(j\to i|\bar{x})\pi(\bar{x},j,t) \\ & = \sum_{j\neq i}\hat{p}(j\to i|\bar{x})\psi(\bar{x},j,t) \quad .\end{aligned} \tag{8}$$

The evolution of this density is straightforwardly deduced from Eqs. (4) and (8):

$$\begin{aligned}\varphi(\bar{x},i,t) = & \sum_{j\neq i}\hat{p}(j\to i|\bar{x})\int \pi(\bar{u},j,o)\delta(\bar{x}-\bar{g}_j(t,\bar{u})) \\ & \times \lambda_j(\bar{x})e^{-\int_o^t\lambda_j(\bar{g}_j(s,\bar{u}))\,ds}\,d\bar{u} \\ & + \sum_{j\neq i}\int_o^t d\tau\int d\bar{u}\,\varphi(\bar{u},j,\tau)\lambda_j(\bar{x}) \\ & \times e^{-\int_o^{t-\tau}\lambda_j(\bar{g}_j(s,\bar{u}))\,ds}\,\hat{p}(j\to i|\bar{x}) \\ & \times \delta(\bar{x}-\bar{g}_j(t-\tau,\bar{u})) \quad .\end{aligned} \tag{9}$$

Both outgoing and ingoing densities are directly re-
2 lated to the pdf $\pi(\bar{x}, i, t)$ as

$$\pi(\bar{x},i,t) = \int \pi(\bar{u},i,o)\delta(\bar{x}-\bar{g}_i(t,\bar{u}))e^{-\int_o^t \lambda_i(\bar{g}_i(s,\bar{u}))\,ds}\,d\bar{u}$$
$$+ \sum_{j\neq i}\int_o^t d\tau \int d\bar{u}\,\psi(\bar{u},j,\tau)\hat{p}(j\to i|\bar{u})$$
$$\times\,\delta(\bar{x}-\bar{g}_i(t-\tau,\bar{u}))\cdot e^{-\int_o^{t-\tau}\lambda_i(\bar{g}_i(s,\bar{u}))\,ds}$$
$$= \int \pi(\bar{u},i,o)\delta(\bar{x}-\bar{g}_i(t,\bar{u}))e^{-\int_o^t \lambda_i(\bar{g}_i(s,\bar{u}))\,ds}\,d\bar{u}$$
$$+ \int_o^t d\tau \int d\bar{u}\,\varphi(\bar{u},i,\tau)\cdot\delta(\bar{x}-\bar{g}_i(t-\tau,\bar{u}))$$
$$\times\, e^{-\int_o^{t-\tau}\lambda_i(\bar{g}_i(s,\bar{u}))\,ds} \quad . \tag{10}$$

II.B. Releasing the Purely Markovian Assumption

The Markovian assumption we have used so far amounts to assuming the system is without memory: No matter how long a system has been evolving in the current configuration, the probability of leaving it after a given time delay remains the same. In other words, the future evolution depends only on the current situation of the plant and not on its past history in the course of the transient. Mathematically speaking, the stochastic branching process is permanently *regenerated*, and the distribution of the sojourn time in a configuration is exponential, with a transition rate having no explicit dependence on time.

As mentioned before, such an assumption is not appropriate for the modeling of human actions. This is one of the main reasons why a semi-Markovian extension of the TPD was soon proposed. In such a stochastic process indeed, only the entry times in a new state are regeneration points. Consequently, any modeling in this assumption must refer to this specific event and then allow any type of distribution for the sojourn time in a system configuration. This can be done by using the ingoing densities introduced above and by slightly modifying Eq. (9):

$$\varphi(\bar{x},i,t) = \sum_{j\neq i}\int_o^t d\tau \int d\bar{u}\,[\pi(\bar{u},j,\tau)\delta(\tau)+\varphi(\bar{u},j,\tau)]$$
$$\times\,\delta(\bar{x}-\bar{g}_j(t-\tau,\bar{u}))q_{ji}(t-\tau;\bar{u}) \quad . \tag{11}$$

Equation (11) makes use of $q_{ji}(t;\bar{u})$, probability per unit time of a transition between dynamics j and i at time t after entering dynamics j at point $\bar{u}$. Comparing Eqs. (9) and (11), one can easily check that this quantity takes the following form in the Markovian case:

$$q_{ji}(t;\bar{u})\delta(\bar{x}-\bar{g}_j(t,\bar{u}))$$
$$= \hat{p}(j\to i|\bar{x})\cdot f_j(t;\bar{u})\cdot\delta(\bar{x}-\bar{g}_j(t,\bar{u})) \quad , \tag{12}$$

where $f_j(t;\bar{u})$ is the pdf of the plant leaving configuration j at time t after entering it at point $\bar{u}$. This density is exponential in the Markovian assumption, but the factorization of q_{ji} in factors f and $\hat{p}$ is valid in other cases, provided the probabilities $\hat{p}(j \to i|\bar{x})$ are (explicitly) independent of the time elapsed in dynamics j (see Appendix A). When this assumption is satisfied, the sojourn time in a given configuration and the transition probabilities out of it are uncoupled, and the modeling can rest on the outgoing density $\psi(\bar{x}, i, t)$. We will nonetheless
present in Secs. XXX a more general treatment based on 3
the ingoing density.

A backward formalism is useful for the deduction of evolution equations for the dynamic equivalent to well-known system characteristics, such as reliability or mean time to failure if the system failure is defined by the first crossing of the border ∂D of a safety domain D in the process variables space. The backward semi-Markov form of the TPD is provided in Appendix B as well as the backward version of all subsequent developments.

III. NEW PSA FEATURES IN SEMI-MARKOV TPD

The main characteristics of the TPD that we summarized in Sec. II are based on the following description of accident sequences. Each time an event causes a modification in the system configuration, the evolution laws of the process variables are likely to be affected. The new dynamics is assumed to start from the final situation reached in the previous configuration, which is reached immediately after the branching is solicited. This double hypothesis is not always true as either an instantaneous change of the process variables value can sometimes occur at the branching point or the occurrence of the event causing the branching is delayed. The corresponding adaptation of the dynamic reliability methodology is presented in Secs. III.A and III.B.

III.A. Extension to Random Shocks

Consider for instance a hydrogen laminar deflagration in the reactor containment in the propagation of an accident. The timescale on which this event takes place is much smaller than the characteristic time on which the whole sequence develops, and the explosion can therefore be assumed instantaneous when modeling the dynamic behavior of the system in these accident conditions. Process variables like H_2, H_2O, CO, and CO_2 concentrations, or containment pressure and temperature, leave the explosion with values different from those they had when the combustion was initiated. The magnitude of the change is mainly driven by a parameter called "burn completeness," which can be expressed either through a correlation on the gas concentrations[16] or via probability density.[17] This implies that these instantaneous changes can be driven by random parameters.

In order to model this aspect, we present in this section a slightly revised approach of an idea propounded in Ref. 18 in order to model random loads within the frame of dynamic reliability. This original work was targeted on structural reliability, which can also be of interest in level-2 PSA problems, but the potential applicability of the model is broader.

Random loads, due to external events, earthquakes etc., are likely to cause random changes of physical parameters of the system, leading possibly to "instantaneous" random changes of some process variables values. The suggested extension models the effect of the random loads during the transitions induced by external events in the following way. A vector $\bar{z}$ of random variables (the burn completeness in our combustion example) is introduced to describe the impact of the shock. Vector $\bar{z}$ is distributed according to the pdf $\phi_{ji}(\bar{z})$ associated with the transition from dynamics j to dynamics i. The process variables are affected by this shock as follows:

$$\bar{x}^{+} = \bar{y}_{ji}(\bar{x}^{-}, \bar{z}) \ , \qquad (13)$$

where $\bar{x}^{+}$ and $\bar{x}^{-}$ are the values of the process variables after and before the transition, respectively. As this change is associated with the branching event, the semi-Markov assumption must be adopted to model it. Introducing the impact of the random shock in Eq. (11), we obtain

$$\begin{aligned}\varphi(\bar{x}, i, t) = \sum_{j \neq i} \int d\bar{x}^{*} \int d\bar{z}\, \delta(\bar{x} - \bar{y}_{ji}(\bar{x}^{*}, \bar{z}))\phi_{ji}(\bar{z}) \\ \times \int_{o}^{t} d\tau \int d\bar{u}\, q_{ji}(t-\tau; \bar{u}) \\ \times \delta(\bar{x}^{*} - \bar{g}_{j}(t-\tau, \bar{u})) \\ \times [\pi(\bar{u}, j, \tau)\delta(\tau) + \varphi(\bar{u}, j, \tau)] \ . \qquad (14)\end{aligned}$$

Equation (14) embodies the following situation. The system entered configuration j at time τ with process variables $\bar{u}$. A time delay $t - \tau$ later, a transition between dynamics j and i occurs, while the continuous variables have reached a value $\bar{x}^{*}$. As a result of a shock situation associated with this transition, and characterized by shock variables $\bar{z}$, an instantaneous change in the value of the process variables takes place, from $\bar{x}^{*}$ to $\bar{x} = \bar{y}_{ji}(\bar{x}^{*}, \bar{z})$. In the absence of a shock, the value of vector $\bar{z}$ is irrelevant, and $\bar{x} = \bar{y}_{ji}(\bar{x}^{*}, \bar{z}) = \bar{x}^{*}$. Straightforward integrations over $\bar{z}$ and $\bar{x}^{*}$ reduce Eq. (14) to Eq. (11).

Reference 18 suggests that the shock variables could also influence the transition probabilities. However, the same vector $\bar{z}$ cannot affect the sojourn time in configuration j and at the same time be distributed according to a pdf depending on the peculiar transition following the system stay in j, as propounded in Ref. 18. Instead, we can assume that the shock variables are system parameters whose values are likely to change randomly after a transition. Therefore, if $\varphi(\bar{x}, i, t, \bar{z})$ is the ingoing density into i at t with process variables $\bar{x}$ and shock variables $\bar{z}$, and if $\phi_{ij}(\bar{z}|\bar{z}^{*})$ is the pdf of the shock variables resulting from the transition $i \to j$, given the latter was entered with $\bar{z}^{*}$, we can write

$$\begin{aligned}\varphi(\bar{x}, i, t, \bar{z}) = \sum_{j \neq i} \int d\bar{x}^{*} \int d\bar{z}^{*}\, \delta(\bar{x} - \bar{y}_{ji}(\bar{x}^{*}, \bar{z}))\phi_{ji}(\bar{z}|\bar{z}^{*}) \\ \times \int_{o}^{t} d\tau \int d\bar{u}\, q_{ji}(t-\tau; \bar{u}, \bar{z}^{*}) \\ \times \delta(\bar{x}^{*} - \bar{g}_{j}(t-\tau, \bar{u})) \\ \times [\pi(\bar{u}, j; \tau)\delta(\tau)\delta(\bar{z}^{*} - \bar{z}_{o}) \\ + \varphi(\bar{u}, j, \tau, \bar{z}^{*})] \ , \qquad (15)\end{aligned}$$

where a dependence on $\bar{z}$ has been added in the probability per unit time q_{ji} and $\bar{z}_{o}$ denotes the initial value of the shock parameters. We have further assumed that the instantaneous change of the process variables after the transition was driven by the shock variables resulting from this transition. This hypothesis is then coherent with the treatment given in Eq. (14).

The addition of the random loads in the dynamic reliability framework mainly appears as averaging the general semi-Markovian equation of the ingoing density over the shock variables distribution. Therefore, to keep the new developments of Secs. XXX more readable, we 3
will skip this potential dependence on $\bar{z}$ in the sequel of the paper.

III.B. Stimulus Activation and Delays in the Branching Process

III.B.1. The Concept of Stimulus

In Sec. II, we summarized, and somewhat reformalized, the fundamental equations of the TPD when the change in the dynamic behavior of the plant occurs with no delay after a solicitation causing a branching in the continuous event tree. In actuality, time delays must often be considered between the triggering of a branching event and its realization. We give the general name of "stimulus" to any situation that can initiate a branching process. Examples of stimuli are, among others,

- the entry in a plant configuration for the possible failure in operation of a subsystem
- the crossing of a setpoint, actuating a control/protection device, or forcing the operator to intervene
- the entry of the system in a given region of the process variable space, corresponding for instance

to the satisfaction of ignition conditions for a gas explosion

- the fulfillment of a necessary condition for the occurrence of an event
- the loss of safety margins to necessary conditions for damage as, for instance, conditions that degrade safety barriers.

As observed from this nonexhaustive list, stimuli can present very different natures, including phenomenological events, as well as control-driven actions. Once a stimulus is activated, a time delay must elapse before the actual occurrence of the branching event. Referring to the examples of stimuli given above, these delays can be

- the time to failure of a piece of equipment from the last branching time
- the time to actuation of a protection device, e.g., due to mechanical inertia, or the diagnostic time taken by an operator before performing an action
- the delay before the appearance of a spark triggering the explosion once the ignition criteria are satisfied
- the time elapsed between the occurrence of a conditioning event and that of the conditioned event
- a grace delay before the actual damage occurs and during which other events like protection actions can be expected.

These times are usually random.

Random delays of action are accounted for in Ref. 4 to model for instance the operator's behavior after the system has crossed a setpoint. The Ref. 4 treatment is based on the assumption that the dispersion of these delays about their mean value was small. But, the main hypothesis consists of performing this uncertainty analysis on the paths of the continuous event tree that had been identified with no delays (or no variability in the delays) in the transitions. This amounts to limiting the competition between events to the comparison of the times necessary to reach the different setpoints respectively associated with them, no matter how long the delays are.

When considering the branching time as the sum of a time to stimulus activation and a delay, competing events are dealt with in a more satisfactory fashion. Indeed, from a given point in the event tree development, the next branching is associated with the event displaying the smallest total time until its occurrence, i.e., the time to the activation of the corresponding stimulus plus the subsequent delay. In the peculiar case of two setpoint transitions in competition, the first setpoint to be crossed could be followed by a rather long delay, enabling the plant's representative trajectory in the process variables space to reach the second threshold, associated with a possibly much shorter delay, which could elapse before the actual occurrence of the first event. This shows how this "stimulus activation + time delay" concept might affect the ordering of events in the tree. Dealing with this kind of competing effect continues as a major source of inspiration for dynamic reliability developments.

III.B.2. Semi-Markov Treatment of Stimulus-Driven Branchings

Let $\mathcal{F}$ be the set of stimuli to be accounted for in the plant evolution following the occurrence of a given initiating event. We denote by $f_i^F(t;\bar{u})$ the pdf of activating the particular stimulus $F \in \mathcal{F}$ after a time interval t spent in configuration i, which was entered at point $\bar{u}$. This dependence on t and $\bar{u}$ is quite general, and setpoints or regions in the process variables space that are associated with stimuli can be modeled via a dependence on $\bar{g}_i(t,\bar{u})$. We also define $h_{ij}^F(t;\bar{u})$, probability per unit time of having a time delay t before a transition between dynamics i and j, if stimulus F was activated at point $\bar{u}$, and $h_i^F(t;\bar{u}) = \sum_{j\neq i} h_{ij}^F(t;\bar{u})$, pdf of the delay before leaving dynamics i in the same conditions.

A transition between plant configurations i and j, through the event induced by stimulus F, will occur at a time between t_F and $t_F + dt_F$ after entering i with a probability

$$dt_F \int_o^{t_F} f_i^F(\tau;\bar{u}) h_{ij}^F(t_F-\tau;\bar{g}_i(\tau,\bar{u}))\,d\tau$$

if no stimulus other than F comes into play (see Fig. 1). When releasing the latter restriction, we can define a time τ_G associated with the occurrence of the event induced by each stimulus $G \in \mathcal{F}$. Then, the event associated with F will cause the branching from i to j in a time interval dt about t if t_F lies in $[t, t+dt]$ *and* if $t_G > t_F$ for all stimuli $G \neq F$ and all transitions out of i. The probability $q_{ij}^F(t;\bar{u})\,dt$ of this situation is then such that

$$q_{ij}^F(t;\bar{u}) = \int_o^t f_i^F(\tau;\bar{u}) h_{ij}^F(t-\tau;\bar{g}_i(\tau,\bar{u}))\,d\tau \times \prod_{G\neq F}\left[1-\int_o^t dt' \int_o^{t'} d\tau\, f_i^G(\tau;\bar{u}) \times h_i^G(t'-\tau;\bar{g}_i(\tau,\bar{u}))\right] . \qquad (16)$$

To illustrate these concepts, let us consider some particular cases:

1. We assume that all stimuli correspond to setpoints and that the distributions of the delays are independent of the process variables. Let $\tau_i^G(\bar{u})$ be the time necessary to reach the setpoint associated with stimulus G in dynamics i and starting from $\bar{u}$. We can then write

$$f_i^G(t;\bar{u}) = \delta(t-\tau_i^G(\bar{u})) \quad \forall G \qquad (17)$$

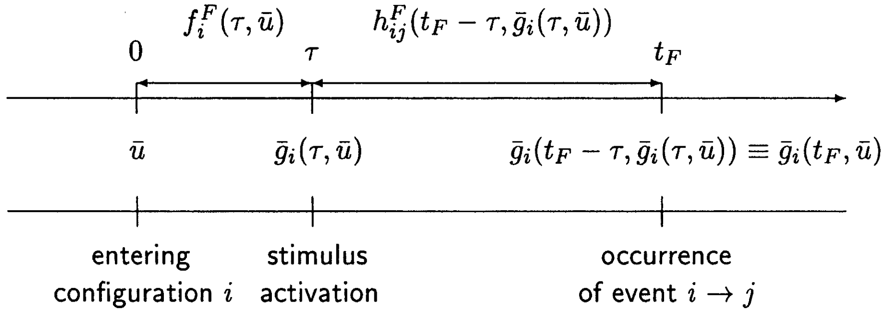

Fig. 1. Two-phase occurrence of the F-induced event.

and thus

$$q_{ij}^F(t;\bar{u}) = h_{ij}^F(t-\tau_i^F(\bar{u})) \times \prod_{G\neq F}[1-H_i^G(t-\tau_i^G(\bar{u}))\cdot\theta(t-\tau_i^G(\bar{u}))] \ , \tag{18}$$

4 where $H_i^G(t)$ is the cumulative density function (cdf) of the delay associated with stimulus G in dynamics i and $\theta(t)$ is the Heaviside stepfunction. Expression (18) highlights the competition between the delays following the activation of the stimuli after deterministic time intervals. The F transition between i and j will take place in $[t, t+dt]$ only if the delays associated with all the transitions corresponding to the other stimuli lead to larger sojourn times in configuration i.

2. A second example that can be envisioned is that of a protection device whose operation is solicited when a setpoint is crossed (stimulus F) and that presents a probability p_f to fail on demand at this time. This situation can be modeled by assuming that a fraction p_f of the stimulus activation probability is rejected at a time t_{AD}, which is an upper bound of the accident duration on which the PSA has to be performed. Equation (17) then becomes

$$f_i^F(t;\bar{u}) = (1-p_f)\cdot\delta(t-\tau_i^F(\bar{u})) + p_f\,\delta(t-t_{AD}) \ . \tag{19}$$

By doing so, pdf $f_i^F(t;\bar{u})$ stays normalized, but the stimulus activation can occur only with a probability $1-p_f$ within the "mission time" of the PSA.

3. Assume finally that experiments related to the occurrence of a phenomena show it is conditioned by the activation of a stimulus F and provide activation probabilities $q_{M_j}^F$ in regions $M_j, j = 1\ldots n$, partitioning the phase-space. Or, consider an operator having a probability $q_{M_j}^F$ to diagnose a problem F when the system lies within region M_j, the delay corresponding to his/her time to action after diagnosis. How can the pdf of the activation time in dynamics i be built in such cases? Let $\tau_{ij}^F(\bar{u})$ be the time required, while evolving in dynamics i from $\bar{u}$, to reach the border of the j'th region M_j visited by the system trajectory in the process variables space, given these regions are ranked in the order they are entered along the process variables evolution in dynamics i. If we assume that stimulus F is instantaneously activated when entering a region, we can write

$$f_i^F(t;\bar{u}) = q_{M_1}^F\delta(t-\tau_{i1}^F(\bar{u})) + (1-q_{M_1}^F)q_{M_2}^F\delta(t-\tau_{i2}^F(\bar{u})) + (1-q_{M_1}^F)(1-q_{M_2}^F)q_{M_3}^F\delta(t-\tau_{i3}^F(\bar{u})) + \cdots + \prod_{j=1}^n(1-q_{M_j}^F)\cdot\delta(t-t_{AD}) \tag{20}$$

with $\tau_{i,j+1}^F(\bar{u}) > \tau_{ij}^F(\bar{u})$, $\forall j$. One can observe that this expression is a generalization of the two previous cases given in Eqs. (17) and (19). A more realistic modeling could consist in taking a uniform distribution within each region. Setting $\Delta\tau_{ij}^F(\bar{u}) = \tau_{i,j+1}^F(\bar{u}) - \tau_{ij}^F(\bar{u})$, we have

$$f_i^F(t;\bar{u}) = \frac{q_{M_1}^F}{\Delta\tau_{i1}^F(\bar{u})}\theta(t-\tau_{i1}^F(\bar{u}))\cdot\theta(\tau_{i2}^F(\bar{u})-t) + \frac{(1-q_{M_1}^F)q_{M_2}^F}{\Delta\tau_{i2}^F(\bar{u})}\theta(t-\tau_{i2}^F(\bar{u}))\theta(\tau_{i3}^F(\bar{u})-t) + \cdots + \prod_{j=1}^n(1-q_{M_j}^F)\cdot\delta(t-t_{AD}) \ . \tag{21}$$

The important particular case of stimuli activated at setpoints and within given regions of phase-space is the object of our companion paper.[11]

Accounting for Eq. (16), Eq. (11) for the ingoing density in configuration i then becomes

$$\varphi(\bar{x},i,t) = \sum_F\sum_{j\neq i}\int_o^t d\tau\int d\bar{u}\,[\pi(\bar{u},j,\tau)\delta(\tau)+\varphi(\bar{u},j,\tau)] \times\delta(\bar{x}-\bar{g}_j(t-\tau,\bar{u}))q_{ji}^F(t-\tau;\bar{u}) \ , \tag{22}$$

while Eq. (10) is modified in the following way:

$$\pi(\bar{x},i,t) = \int_o^t d\tau \int d\bar{u}\,[\pi(\bar{u},i,\tau)\delta(\tau) + \varphi(\bar{u},i,\tau)] \times \delta(\bar{x} - \bar{g}_i(t-\tau,\bar{u}))\cdot(1 - P_i(t-\tau;\bar{u})) \ , \quad (23)$$

where

$$P_i(t;\bar{u}) \equiv \sum_F \int_o^t p_i^F(\tau;\bar{u})\,d\tau \equiv \sum_F \sum_{j\neq i} \int_o^t q_{ij}^F(\tau;\bar{u})\,d\tau \ . \quad (24)$$

5 Though the mathematical formulation of Eqs. XXX is somewhat heavy, they are nothing but a direct transcription of the probability of the different random processes in competition.

Some simplification in the interpretation of these developments can be obtained if the probability per unit time q_{ji}^F can be factorized, as mentioned in Sec. II.B, and the outgoing density can consequently be used. The corresponding evolution equations are given in Appendix C.

III.B.3. Remarks

The preceding developments have allowed us to account for the delay following the activation of a stimulus before the actual occurrence of the event's inducing a change of dynamics. This transition time appears thus as the sum of two random times, hence, the convolution products introduced from Eq. (16) on. The competition between events is also embodied by Eq. (16), where the process corresponding to the shortest total transition time (activation + delay) is the one responsible for driving the system toward a new dynamic evolution. Let us however mention that the semi-Markovian framework that has been used induces the following consequence: The entry in a new dynamics is a regeneration point for the stochastic process describing the system evolution. In practice, this means that all stimuli that are activated at the time of the transition are disactivated once the new configuration is entered. Releasing this limitation is discussed in Sec. IV.

One could also wonder if a stochastic modeling for the activation time of the stimuli is mandatory. In many practical situations indeed, this time is deterministic, be it the entry in a new dynamics or the crossing of a setpoint while following a given dynamic trajectory (see examples in Sec. III.B.2). Even if there is some possible randomness in the position of a setpoint, this could be accounted for in the distribution of the corresponding delay. Though the pdf's f_F^i will often reduce to a Dirac peak, thereby bringing some simplifications in the expressions above [see, e.g., Eq. (18)], we chose to keep the developments as general as possible in the theoretical perspective of this paper. A positive side effect of this choice is to better enlighten the competing process among all events likely to cause the dynamics to change [see Eq. (16)]. Another advantage of keeping a fully probabilistic description of the branching process appears in a companion paper[11]: A partition in cells of the region of interest in the process variables space leads to a probabilistic interpretation of the proportion of dynamic trajectories going from one cell to another, even when trajectories correspond to Dirac peaks.

To conclude these remarks, let us notice that all stimuli are likely to be activated in any dynamics in our present framework. In actuality, some stimuli could be specific to a given set of configurations. If stimulus F cannot be activated in dynamics i, this case can simply be accounted for by considering a pdf $f_i^F(t;\bar{u}) = \delta(t - t_{AD})$, where t_{AD} is an upper bound of the accident duration for the transient under study.

IV. INCOMPLETE DISACTIVATION OF THE STIMULI AFTER A TRANSITION

IV.A. Stating the Problem

As mentioned before, the main consequence of the semi-Markovian framework we have adopted up to now is the "regeneration" (i.e., disactivation) of all stimuli as soon as a new dynamics is followed.

This assumption does not always hold. Indeed, some events are due to occur some time after the activation of the corresponding stimulus no matter which state the system lies in. Consider for instance the following situation: A setpoint crossing has triggered the stimulus for the intervention of a safeguard system, but an unrelated hardware failure, not affecting the safeguard itself, provokes a change of configuration before the end of the delay associated with the protection device actuation. The dynamics is clearly modified, but without preventing the protection action from taking place soon after. The corresponding activated stimulus was therefore unaffected by the hardware failure. The change in dynamics between the activation of the stimulus and the actual occurrence of the event is only likely to have altered the distribution of the time delay. Other practical examples of stimuli keeping activated after a change of dynamics are given in the illustrative application treated in Sec. V.

This means that the semi-Markovian restriction in our previous developments has to be left aside in order to include such circumstances in the theory. Let us then try to formalize the problem. Let $\mathcal{S}(\ni F)$ be the subset of all stimuli that have been activated at the time the current dynamics is left because of the occurrence of the event induced by stimulus F. The latter is of course disactivated as well as some of (but not necessarily all) the stimuli belonging to $\mathcal{S}/\{F\}$. In the new dynamics, the stimuli that were disactivated could be treated as before since they were "regenerated" by the transition. As for the stimuli that have stayed activated, and that form a subset $\mathcal{A}$ of $\mathcal{S}$, the distribution of the remaining time delay before the occurrence of the event they condition should now be considered. This implies that some information on the history of the system since their activation should be kept in memory. We then see that the problem becomes non-Markovian as soon as at least one of the stimuli in $\mathcal{S}/\{F\}$ keeps activated after the change in dynamics.

IV.B. Conditional Density of the Residual Delay

Let us first establish the conditional pdf $\tilde{h}_i^F$ of the delay following the activation of stimulus F, given that the latter occurred before the entry in configuration i and given F remained activated after the transition to i.

Let τ be the time at which configuration i was entered by the system, with process variables $\bar{u}$, and $\tau_F < \tau$ the time at which stimulus $F \in \mathcal{A}$ was activated.

We will consider different situations corresponding to different types of delays that can be envisioned. We will first concentrate on the simple particular case where the delay density depends only on time. Let us then first build the unconditional pdf h^F of the delay associated with a stimulus F, activated at τ_F in configuration j, with a transition to configuration i occurring at time τ. We have

$$h^F(t-\tau_F; j \to i, \tau) = \begin{cases} h_j^F(t-\tau_F) & \text{if } \tau_F \le t \le \tau \\ (1-H_j^F(\tau-\tau_F)) \cdot \dfrac{h_i^F(t-\tau_F)}{1-H_i^F(\tau-\tau_F)} & \\ \qquad \text{if } t \ge \tau \ . & \end{cases} \tag{25}$$

The second line of Eq. (25) is deduced from the following reasoning. The delay will be greater than $\tau - \tau_F$ if it is not elapsed in configuration j, i.e., with a probability $1 - H_j^F(\tau - \tau_F)$; the pdf of the delay in dynamics i has then to be used, conditional to the fact that the delay is greater than $\tau - \tau_F$. Writing this expression is possible if one assumes that the transition from j to i does not modify the reference time for the delay, i.e., that the delay elapsing is considered from τ_F in both dynamics j and i.

Considering now the conditional pdf $\tilde{h}_i^F$ of the residual delay associated with stimulus F that remained activated after the transition to dynamics i, we have, setting $\Delta\tau_F = \tau - \tau_F$,

$$\tilde{h}_i^F(\Delta t \mid \Delta\tau_F) = \frac{h^F(t-\tau_F; j \to i, \tau)}{1-H_j^F(\Delta\tau_F)} = \frac{h_i^F(\Delta t + \Delta\tau_F)}{1-H_i^F(\max(0,\Delta\tau_F))} \tag{26}$$

if $\Delta t = t - \tau$ is the time that has elapsed since the entry of the system in configuration i (see Fig. 2). Indeed, the actual delay before the occurrence of the event triggered by the activation of F is $\Delta t + \Delta\tau_F$, but the corresponding distribution must be truncated up to $\Delta\tau_F$; i.e., it must be conditioned to the survival of the stimulus activation during the time interval required to enter the new configuration i. A general expression of the conditional density has been given in Eq. (26), also valid for the case where $\tau < \tau_F$, for which the original delay pdf is obtained. As activations before and after the last change of dynamics are to be mixed in our problem, this conditional density function can thus be used in both cases. It will always be considered in the sequel of this section.

However, before considering more complicated dependences of the delay pdf, we can shortly discuss the hypothesis of time continuity of the delay, before and after the change of dynamics, that underlines the developments made so far. Indeed, assuming that the delay keeps elapsing in the same way after the change of dynamics, while the pdf is modified, could be questionable.

An even stronger assumption would be the renewal of the delay elapsing; i.e., the reference time for the delay after the transition is the transition time itself. In such a case, there is no need anymore to envision a non-Markov treatment of the problem, as any stimulus that remains activated after a transition can be considered disactivated, and immediately reactivated after the transition.

Another approach would then consist of assuming that the delay will elapse in dynamics i, based on the hypothesis that the probability of having the delay elapsed

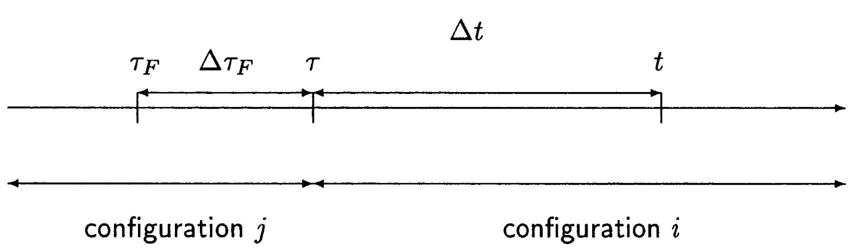

Fig. 2. Time line for the residual delay after a transition.

must be continuous before and after the transition. In other words, the reference time after the transition should no longer be τ_F, but $\tilde{\tau}_F$, in such a way that $H_j^F(\tau - \tau_F) = H_i^F(\tau - \tilde{\tau}_F)$. This amounts to assuming that the total probability of a delay longer than the time interval $\tau - \tau_F$ up to the transition is unaffected by this transition.

If the delay pdf is unaltered by the transition, we will have of course $\tau_F = \tilde{\tau}_F$. In general terms, however, we will have, instead of Eq. (25),

$$h^F(t - \tau_F; j \to i, \tau) = \begin{cases} h_j^F(t - \tau_F) & \text{if } \tau_F \le t \le \tau \\ (1 - H_j^F(\tau - \tau_F)) \cdot \dfrac{h_i^F(t - \tilde{\tau}_F)}{1 - H_i^F(\tau - \tilde{\tau}_F)} & \\ \quad = h_i^F(t - \tilde{\tau}_F) & \text{if } t \ge \tau \end{cases} \tag{27}$$

with

$$\tilde{\tau}_F = \tau - (H_i^F)^{-1}(H_j^F(\tau - \tau_F)) \tag{28}$$

with $(H_i^F)^{-1}$ denoting the inverse of function $H_i^F(t)$.

We will however keep the assumption of the time continuity in the sequel of this section.

A second situation of interest corresponds to a delay density displaying a time dependence either along a dynamic trajectory or within a given region of the process variables space, i.e., with $h_i^F(t; \bar{u})$. This could for instance be the case in a level-2 accident, when an explosion takes place if the zone where ignition criteria are satisfied is entered and if a delay associated with the occurrence of a spark has to be elapsed before leaving the ignition zone. Then,

$$\tilde{h}_i^F(\Delta t; \bar{u} | \Delta\tau_F) = \frac{h_i^F(\Delta t + \Delta\tau_F; \bar{u}_i^F)}{1 - H_i^F(\max(0, \Delta\tau_F); \bar{u}_i^F)} \ , \tag{29}$$

where $\bar{u}_i^F$ is such that $\bar{u} = \bar{g}_i(\Delta\tau_F, \bar{u}_i^F)$. For $\Delta\tau_F > 0$, vector $\bar{u}_i^F$ is thus not the value of the process variables at the time stimulus F was activated since this occurred in a configuration different from i, associated with a dynamic evolution that is not $\bar{g}_i$. Yet, $\bar{u}_i^F$ appears as a virtual initial condition for the dynamic evolution in configuration i, with respect to τ_F, and leads to the true trajectory after τ.

In a third possible case, the triggered event has a nonzero probability of taking place in a given region of the process variables space, and the delay expresses only the time required to reach a given point in this zone. In order to obtain simpler mathematical expressions, let us assume that the event associated with stimulus F occurs for a value of the process variable p within the range $[p_{min}, p_{max}]$, with a probability density $f(p)$. The stimulus was activated when crossing the setpoint at the lower end p_{min} of the support of $f(p)$. The distribution of the delay in state i relates to $f(p)$ according to

$$h_i^F(\Delta t; \bar{u}) = f(g_{ip}(\Delta t, \bar{u})) \cdot |q_{ip}(\bar{g}_i(\Delta t, \bar{u}))| \tag{30}$$

as long as the p component $g_{ip}(t, \bar{u})$ of the system trajectory is monotonically increasing. In the latter expression, q_{ip} is the time derivative of the p component of vector $\bar{x}$ in dynamics i. If we now assume that state i was entered after a transition that does not disactivate stimulus F, and if the monotonic behavior of p is still observed after the transition, we have for the conditional density of the delay 6 6

$$\tilde{h}_i^F(\Delta t; \bar{u}) = \frac{f(g_{ip}(\Delta t, \bar{u})) \cdot |q_{ip}(\bar{g}_i(\Delta t, \bar{u}))|}{1 - F(g_{ip}(0, \bar{u}))} \ , \tag{31}$$

where $F(p)$ is the cdf associated with $f(p)$ and $g_{ip}(0, \bar{u})$ is the value of p at the transition time. Note that this time the conditional pdf is independent of τ_F since the event induced by F occurs at a given position in the process variables space and not explicitly at a given time. In this case, we can keep the semi-Markov treatment developed in Sec. IV.A provided some adaptations are brought. The stimuli that remain activated after a transition are assumed to be disactivated at the transition time and immediately reactivated after the transition [therefore with a corresponding pdf $f_i^F(t; \bar{u}) = \delta(t)$ in the new state i], with the distribution of the delay being given by Eq. (31) in the new configuration. 7

When the assumption on the monotonic evolution of $g_{ip}(t, \bar{u})$ is released, the situation becomes more complicated, as Eq. (30) is no longer valid as such. The treatment that then needs to be done is similar to that which was performed in Ref. 19 for the determination of a mean estimator of the failure probability in the Monte Carlo simulation of a dynamic reliability problem with distributed safety borders. Assume that $g_{ip}(t, \bar{u})$ is increasing up to a maximum $p^* \in [p_{min}, p_{max}]$. As soon as the value of p starts decreasing after p^*, the probability of the event occurrence vanishes if we suppose it is associated with a first passage at a given value of p. This observation is valid even if no change of dynamics took place. The system has then survived this first entry in the support of $f(p)$ on $[p_{min}, p^*]$ with a probability $1 - F(p^*)$. If the system reenters the support of $f(p)$ later on, it is impossible for the event to actually occur below p^*, again with the assumption that the event occurrence is associated with a first crossing of a value of p. Above this value p^*, the event can then take place at a level in $[p, p + dp]$ with a probability $f(p)\,dp/(1 - F(p^*))$. A simple way of modeling this case consists of the following steps: 6

- disactivation of the stimulus F when the maximum p^* is passed (see Sec. IV.E)
- reactivation of F when p^* (and not p_{min}!) is reached again
- use of a conditional distribution for the delay, similar in form to Eq. (31), but with respect to level p^* instead of p.

IV.C. Non-Markov Treatment of Stimulus-Driven Branchings

IV.C.1. Probability of a Next Event

The semi-Markov framework in Sec. III, and its inherent property of disactivating all activated stimuli each time the dynamics changes, allowed us to model the system evolution while referring only to the entry in new configurations. Once activated stimuli can survive the transitions, one could think of generalizing the "disactivation + instantaneous reactivation" trick that we suggested at the end of the last paragraph: For all stimuli belonging to $\mathcal{A}$, the density of the activation time in the new configuration would reduce to a Dirac peak, while the delay would be distributed according to one of the conditional pdf's presented in Sec. IV.B; these results could then be introduced into Eq. (16), and the semi-Markov approach would be formally conserved. In doing this, however, we could not keep track of the activation times of stimuli that would be activated *after* entering the present configuration and that could remain activated after the next change of dynamics. We therefore have to consider separately two types of events, which are to be handled at the same level, since none of them can any longer regenerate the system:

- the activation of a new stimulus $F \in \mathcal{F}/\mathcal{A}$, before the end of the delay associated with any of the stimuli $G \in \mathcal{A}$ (case 1). After its activation, F is added to $\mathcal{A}$.
- the occurrence of a new stimulus-driven branching before any new activation, with the delay associated with a stimulus of $\mathcal{A}$ being elapsed (case 2).

In the following, τ^* denotes the time of occurrence of the last event, either a change of dynamics ($\tau^* = \tau$, time at which the last configuration change took place at point $\bar{u}$) or a stimulus activation ($\tau^* > \tau$), which took place at $\bar{u}^*$; $\vec{\tau}_{\mathcal{A}}$ is a shortcut notation for the times of activation τ_G of all $G \in \mathcal{A}$. Subset $\mathcal{A}$ has to be enlarged each time case 1 is met and updated in case 2. It must be highlighted that because of case-1 events, $\mathcal{A}$ will contain stimuli that were activated before entering the current configuration and stimuli that were activated afterward. We will however keep the tilded notations for the pdf's of the delays in both cases since in Eqs. (26), (29), and (31), the denominator must not be considered when $\tau_F > \tau$.

In case 1, the probability that F will be the next stimulus to be activated, in configuration i and in $[t, t+dt]$, before any other event occurs, given the subset $\mathcal{A}$ of stimuli having remained activated after entering i, is written

$$p_i^{F*}(t;\tau^*,\tau,\bar{u}^*,\vec{\tau}_{\mathcal{A}},\mathcal{A})\,dt = \frac{f_i^F(t-\tau;\bar{u})\,dt}{1-F_i^F(\tau^*-\tau;\bar{u})} \cdot \prod_{\substack{H\notin\mathcal{A}\\ H\neq F}} \frac{1-F_i^H(t-\tau;\bar{u})}{1-F_i^H(\tau^*-\tau;\bar{u})} \times \prod_{G\in\mathcal{A}} \frac{1-\tilde{H}_i^G(t-\tau;\bar{u}|\Delta\tau_G)}{1-\tilde{H}_i^G(\tau^*-\tau;\bar{u}|\Delta\tau_G)} \ , \tag{32}$$

where $\bar{u}$ is such that $\bar{u}^* = \bar{g}_i(\tau^*-\tau,\bar{u})$. Note that we have willingly kept $\bar{u}^*$ in the arguments of p_i^{F*}, even though $\bar{u}$ is the only value of the process variables appearing on the right side of Eq. (32). The reason of this notation appears more clearly in our companion paper.[11] All activation and delay elapsing processes are to be made conditional to the occurrence of the last event at τ^*, such as in system engineering for non-Markovian components.[20–22] The interpretation of Eq. (32) is straightforward: It is the probability that stimulus F is activated after a time $t-\tau$ in dynamics i, conditionally to $t > \tau^*$, while no other stimulus is activated and while no event induced by an already activated stimulus takes place on $[\tau^*, t]$. We assume that the updating of set $\mathcal{A}$ following the activation of F automatically implies that of vector $\vec{\tau}_{\mathcal{A}}$.

As for case 2, we consider the probability that the event triggered by the activated stimulus F will occur in $[t, t+dt]$ and bring the system in dynamics j, under the same conditions as above:

$$p_{ij}^F(t;\tau^*,\tau,\bar{u}^*,\vec{\tau}_{\mathcal{A}},\mathcal{A})\,dt = \frac{\tilde{h}_{ij}^F(t-\tau;\bar{u}|\Delta\tau_F)\,dt}{1-\tilde{H}_i^F(\tau^*-\tau;\bar{u}|\Delta\tau_F)} \times \prod_{\substack{G\in\mathcal{A}\\ G\neq F}} \frac{1-\tilde{H}_i^G(t-\tau;\bar{u}|\Delta\tau_G)}{1-\tilde{H}_i^G(\tau^*-\tau;\bar{u}|\Delta\tau_G)} \times \prod_{H\notin\mathcal{A}} \frac{1-F_i^H(t-\tau;\bar{u})}{1-F_i^H(\tau^*-\tau;\bar{u})} \ . \tag{33}$$

The reasoning leading to Eq. (33) is similar to the previous one. Note however that the probability per unit time $\tilde{h}_{ij}^F(\tau^*-\tau;\bar{u}|\Delta\tau_F)$ is conditioned by $1-\tilde{H}_i^F(\tau^*-\tau;\bar{u}|\Delta\tau_F)$, i.e., by the complement to the whole cdf. Indeed, the transition to dynamics j in $[t, t+dt]$ is chosen, given no change of dynamics, whatever the next configuration, and was caused by stimulus F before τ^*. Expression (33) must be complemented by a logical operator $\delta_{ij}^F(\mathcal{A} \to \mathcal{A}')$ giving the set of stimuli that keep activated after the transition $i \to j$ due to the F-induced event. In most cases, $\mathcal{A}'$ is a subset of $\mathcal{A}$.

IV.C.2. Evolution Equations

Most bricks necessary to adapt the previous evolution equation of the ingoing density [see Eq. (22)] are now available. Yet, we must still introduce two different ingoing densities, associated with each type of next event that might occur, respectively. Before doing this, we can observe that the future evolution of the system is always conditioned by $n_{\mathcal{A}}+1$ reference times if $n_{\mathcal{A}}$ is the size of set $\mathcal{A}$. These reference times are the activation times of all stimuli belonging to $\mathcal{A}$, and the entry time in the current dynamics. When a new stimulus activation takes place, additional time has to be kept in memory, with all other reference times being unaltered. In case of a change of dynamics, the entry time is of course updated, while some activation times become irrelevant if the corresponding stimuli do not remain activated after the transition.

Let us then define the two ingoing densities that describe the system evolution. We make them explicitly dependent on the current time t, though t has to take the value of one of the reference times mentioned above for the densities not to vanish. We thus have

- $\varphi_{in}(\bar{x}, j, t, \vec{\tau}_{\mathcal{A}}, \mathcal{A})$, ingoing density in dynamics j at point $\bar{x}$ and time t, with a set $\mathcal{A}$ of stimuli activated at $\vec{\tau}_{\mathcal{A}}$ and remaining activated after entering the new configuration j. If we do not consider the possibility of additional activations caused by the transition, t is an upper bound of all the components of vector $\vec{\tau}_{\mathcal{A}}$.

- $\varphi_F(\bar{x}, j, t, \tau, \vec{\tau}_{\mathcal{A}}, \mathcal{A})$, density of activation of stimulus F in dynamics j, at $(\bar{x}, t)$ for an entry in j at time τ, this activation resulting in a set $\mathcal{A}$ of activated stimuli (i.e., $F \in \mathcal{A}$). In this case, t is of course the F component of $\vec{\tau}_{\mathcal{A}}$, and $t > \tau_G, \forall G \in \mathcal{A}$, $G \neq F$. Therefore, mentioning t in addition to $\vec{\tau}_{\mathcal{A}}$ in the arguments of φ_F is redundant, but it is kept to make an explicit reference to the last activation time. Yet, this activation density will have to be used systematically with a Dirac peak $\delta(t-\tau_F)$, when variables t and τ_F are treated independently.

Following the remark made on the inclusion of $t = \tau_F$ in the arguments of φ_F, it must be noticed that both densities have the same dimensions, for a given size of set $\mathcal{A}$, i.e., the inverse of the dimensions of $\bar{x}$, times $t^{-(\#\mathcal{A}+1)}$.

If as before τ denotes the entry time in dynamics i and τ^*, the time at which the last event occurred, we have

$$\varphi_{in}(\bar{x}, j, t, \vec{\tau}_{\mathcal{A}}, \mathcal{A})$$

$$= \sum_{\mathcal{A}'\supset\mathcal{A}} \sum_{F\in\mathcal{A}'} \sum_{i\neq j} \int d\bar{u} \int_o^t d\tau^* \int_o^{\tau^*} d\tau \int_o^{\tau^*} \dots \int_o^{\tau^*} d\vec{\tau}_{\mathcal{A}'/\mathcal{A}}\, \delta(\bar{x} - \bar{g}_i(t-\tau^*, \bar{u}))\, p_{ij}^F(t; \tau^*, \tau, \bar{u}, \vec{\tau}_{\mathcal{A}'}, \mathcal{A}')$$

$$\times \left[\varphi_{in}(\bar{u}, i, \tau^*, \vec{\tau}_{\mathcal{A}'}, \mathcal{A}')\delta(\tau^* - \tau) + \sum_{G\in\mathcal{A}'} \varphi_G(\bar{u}, i, \tau^*, \tau, \vec{\tau}_{\mathcal{A}'}, \mathcal{A}')\delta(\tau^* - \tau_G) \right] \delta_{ij}^F(\mathcal{A}' \to \mathcal{A}) \ . \tag{34}$$

Indeed, the entry in configuration j is possible from any configuration i in which an already activated stimulus F can induce the transition $i \to j$ either if the previous event was the entry in i or if it was any of the stimulus activations that took place in i. Vector $\vec{\tau}_{\mathcal{A}}$ is updated by conserving the components of $\vec{\tau}_{\mathcal{A}'}$ corresponding to the stimuli belonging to $\mathcal{A}$. Since the actual activation times of the stimuli belonging to $\mathcal{A}'/\mathcal{A}$ do not need to be accounted for after the change of dynamics, the contributions to $\varphi_{in}(\bar{x}, j, t, \vec{\tau}_{\mathcal{A}}, \mathcal{A})$ have to be summed up on all possible values of the components of $\vec{\tau}_{\mathcal{A}'/\mathcal{A}}$, hence this multiple integral. We then see the dependence of the dimensions of the ingoing density on the size of set $\mathcal{A}$.

As for the density of activation of F, we can write

$$\varphi_F(\bar{x}, j, t, \tau, \vec{\tau}_{\mathcal{A}+\{F\}}, \mathcal{A}+\{F\})$$

$$= \int d\bar{u} \int_\tau^t d\tau^*\, \delta(\bar{x} - \bar{g}_j(t-\tau^*, \bar{u}))\, p_j^{F*}(t; \tau^*, \tau, \bar{u}, \vec{\tau}_{\mathcal{A}}, \mathcal{A})$$

$$\times \left[[\pi(\bar{u}, j, \tau)\delta(\tau)\delta_{\mathcal{A},\varnothing} + \varphi_{in}(\bar{u}, j, \tau^*, \vec{\tau}_{\mathcal{A}}, \mathcal{A})]\delta(\tau^* - \tau) + \sum_{G\in\mathcal{A}} \varphi_G(\bar{u}, j, \tau^*, \tau, \vec{\tau}_{\mathcal{A}}, \mathcal{A})\delta(\tau^* - \tau_G) \right] , \tag{35}$$

where $\varnothing$ is the empty set. The second and third terms in the main brackets on the right side of Eq. (35) lead to an interpretation similar to what has been done for Eq. (34). One should notice however that an activation is always the first event to take place after the transient initiation if we assume that no stimulus is initially activated, hence the first term in these brackets. Vector $\vec{\tau}_{\mathcal{A}}$ is updated by adding to it the activation time $\tau_F \equiv t$.

The probability density in state i then is written

$$\pi(\bar{x}, i, t; \mathcal{A}) = \int d\bar{u}^* \int_o^t d\tau^* \int_o^{\tau^*} d\tau \int_o^{\tau^*} \ldots \int_o^{\tau^*} d\vec{\tau}_{\mathcal{A}}\, \delta(\bar{x} - \bar{g}_i(t-\tau^*, \bar{u}^*))(1 - P_i(t;\tau^*,\tau,\bar{u}^*,\vec{\tau}_{\mathcal{A}},\mathcal{A}))$$
$$\times \left[[\pi(\bar{u}^*, j, \tau)\delta(\tau)\delta_{\mathcal{A},\varnothing} + \varphi_{in}(\bar{u}^*, i, \tau^*, \vec{\tau}_{\mathcal{A}}, \mathcal{A})]\delta(\tau^* - \tau) + \sum_{F\in\mathcal{A}} \varphi_F(\bar{u}^*, i, \tau^*, \tau, \vec{\tau}_{\mathcal{A}}, \mathcal{A})\delta(\tau^* - \tau_F) \right] , \quad (36)$$

where $P^i(t;\tau^*,\tau,\bar{u}^*,\vec{\tau}_{\mathcal{A}},\mathcal{A})$ is the probability that the next event in dynamics i will occur before t, provided the last event took place at $(\tau^*, \bar{u}^*)$, dynamics i was entered at τ, and the stimuli belonging to $\mathcal{A}$ were activated at $\vec{\tau}_{\mathcal{A}}$. We have

$$1 - P_i(t;\tau^*,\tau,\bar{u}^*,\vec{\tau}_{\mathcal{A}},\mathcal{A}) = \prod_{G\in\mathcal{A}} \frac{1 - \tilde{H}_i^G(t-\tau;\bar{u}|\Delta\tau_G)}{1 - \tilde{H}_i^G(\tau^*-\tau;\bar{u}|\Delta\tau_G)} \times \prod_{H\notin\mathcal{A}} \frac{1 - F_i^H(t-\tau;\bar{u})}{1 - F_i^H(\tau^*-\tau;\bar{u})} , \quad (37)$$

where again $\bar{u}^* = \bar{g}_i(\tau^* - \tau, \bar{u})$.

IV.D. Disactivation Rules and Random Shocks

Section III.A presented how to model instantaneous changes in the value of process variables that can take place at a transition time between two dynamics. As mentioned in Sec. III.A, the methodology roughly amounts to averaging the evolution equations of the problem on the distribution of random shock variables determining the magnitude of the jump in the process variables value. For this reason, coupling this feature with the stimulus-driven branching process is direct.

However, this instantaneous modification of the process variables value is likely to bring the system on the other side of a setpoint corresponding to a stimulus or within a region where a stimulus can be activated. The system could also exit a region where a stimulus remains
7 activated (see Sec. IV.E). The occurrence of these situations depends on the value of the shock variables associated with the jump.

In the semi-Markov model, some stimuli can then be directly activated when entering the new dynamics, depending on the magnitude of the random jump undergone by the continuous variables. In the non-Markov case, the logical functions $\delta_{ij}^F(\mathcal{A} \to \mathcal{A}')$ are to be made dependent on the value of the process variables before and after the jump, or alternatively on the shock variables and the process variables before this jump.

IV.E. Disactivation of a Stimulus Without Change of Dynamics

Until now, we have assumed that activated stimuli could only be disactivated when a change of dynamics took place. This situation is forced to occur in the semi-Markov framework, while disactivation rules are to be introduced in the non-Markov treatment to define which stimuli keep activated after a transition between system states.

However, stimuli are likely to be disactivated once the process variables cross a setpoint or enter a given region of phase-space, even though the system keeps evolving in the same dynamics. Such a circumstance can be encountered when modeling combustion processes in PSA2 analyses. The combustion stimulus is activated when entering a flammability region, but no ignition will take place if the delay following activation is larger than the time taken by the system to exit the flammability zone.

Having now developed the non-Markov theory, we can propound a way of modeling the disactivation of stimulus F. Let $\bar{F}$ be a fictitious stimulus associated with F and assume the pdf of activation of $\bar{F}$ is identical to that of disactivation of F (note that this last pdf cannot be handled in our theory). In the combustion example mentioned above, $\bar{F}$ will be activated when the system goes out of the flammability region. A nil delay is associated with the $\bar{F}$ event that we still need to define. If i is the current dynamics, this event is a transition from i to itself. In other words, the activation of $\bar{F}$ will be instantaneously followed by this transition from i to i.

In the non-Markov treatment, we can associate disactivation rules to this transition. In this case, F and $\bar{F}$ will be disactivated, while all other stimuli that belonged to the set $\mathcal{A}$ before the $\bar{F}$-induced transition remain activated afterward.

IV.F. Consistency of the Stimulus-Driven Approach

Though the final expressions in both forward and backward cases are rather complicated, they are obtained by introducing successively new modeling capabilities in the classical TPD. In order to verify that the well-known theory of dynamic reliability is a particular case of these new developments, we can trace back the different assumptions brought into the theory and show that we find again the classical TPD. This can be done for instance in the forward case, where the use of two ingoing densities makes it not obvious at first glance to check the coherence of the stimulus-driven approach with the C.K. equations.

Let us first assume that no stimulus can remain activated after a change of dynamics. In mathematical words, we have $\delta_{ij}^F(\mathcal{A}' \to \mathcal{A}) = \delta_{\mathcal{A},\varnothing}$, for each transition $i \to j$ induced by each F. This means that the dependence of φ_{in} in $\mathcal{A}$ is irrelevant since it could only depend on the empty set $\varnothing$, and we will no longer mention it. In this case, Eq. (34) becomes

$$\varphi_{in}(\bar{x},j,t) = \sum_{\mathcal{A}} \sum_{F\in\mathcal{A}} \sum_{i\neq j} \int d\bar{u} \int_o^t d\tau^* \int_o^{\tau^*} d\tau \int_o^{\tau^*} \dots \int_o^{\tau^*} d\vec{\tau}_{\mathcal{A}}\, \delta(\bar{x} - \bar{g}_i(t-\tau^*,\bar{u}))$$
$$\times\, p_{ij}^F(t;\tau^*,\tau,\bar{u},\vec{\tau}_{\mathcal{A}},\mathcal{A}) \left[\sum_{G\in\mathcal{A}} \varphi_G(\bar{u},i,\tau^*,\tau,\vec{\tau}_{\mathcal{A}},\mathcal{A})\delta(\tau^* - \tau_G) \right] . \quad (38)$$

Indeed, the entry in a new dynamics asks first for at least one stimulus to be activated, and the contribution of φ_{in} in the integral of Eq. (34) disappears. As for Eq. (35), it now takes the following form:

$$\varphi_F(\bar{x},j,t,\tau,\vec{\tau}_{\mathcal{A}+\{F\}},\mathcal{A}+\{F\})$$
$$= \delta_{\mathcal{A},\varnothing} \int d\bar{u}\, [\pi(\bar{u},j,\tau)\delta(\tau) + \varphi_{in}(\bar{u},j,\tau)]\delta(\bar{x} - \bar{g}_j(t-\tau,\bar{u}))p_j^{F*}(t;\tau,\tau,\bar{u},\vec{\tau}_{\varnothing},\varnothing)$$
$$+ \sum_{G\in\mathcal{A}} \int d\bar{u} \int_\tau^t d\tau^*\, \varphi_G(\bar{u},j,\tau^*,\tau,\vec{\tau}_{\mathcal{A}},\mathcal{A})\delta(\tau^* - \tau_G)\delta(\bar{x} - \bar{g}_j(t-\tau^*,\bar{u}))p_j^{F*}(t;\tau^*,\tau,\bar{u},\vec{\tau}_{\mathcal{A}},\mathcal{A}) . \quad (39)$$

Introducing then Eq. (39) into Eq. (38), we obtain

$$\varphi_{in}(\bar{x},j,t) = \sum_F \sum_{i\neq j} \int d\bar{u} \int_o^t d\tau\, [\pi(\bar{u},i,\tau)\delta(\tau) + \varphi_{in}(\bar{u},i,\tau)]\delta(\bar{x} - \bar{g}_i(t-\tau,\bar{u}))$$
$$\times \left[\int_\tau^t d\tau_F\, p_i^{F*}(\tau_F;\tau,\tau,\bar{u},\vec{\tau}_{\varnothing},\varnothing)p_{ij}^F(t;\tau_F,\tau,\bar{g}_i(\tau_F-\tau,\bar{u}),\tau_F,\{F\}) \right]$$
$$+ \sum_{\mathcal{A}} \sum_{F\in\mathcal{A}} \sum_{i\neq j} \sum_{G\in\mathcal{A}} \sum_{H\in\mathcal{A}/\{G\}} \int d\bar{u} \int_o^t d\tau_G \int_o^{\tau_G} d\tau \int_\tau^{\tau_G} ds \int_o^{\tau_G} \dots \int_o^{\tau_G} d\vec{\tau}_{\mathcal{A}/\{G\}}$$
$$\times\, \varphi_H(\bar{u},i,s,\tau,\vec{\tau}_{\mathcal{A}/\{G\}},\mathcal{A}/\{G\})\delta(s-\tau_H)\delta(\bar{x} - \bar{g}_i(t-s,\bar{u}))$$
$$\times\, p_i^{G*}(\tau_G;s,\tau,\bar{u},\vec{\tau}_{\mathcal{A}/\{G\}},\mathcal{A}/\{G\})p_{ij}^F(t;\tau_G,\tau,\bar{g}_i(\tau_G-s,\bar{u}),\vec{\tau}_{\mathcal{A}},\mathcal{A}) . \quad (40)$$

Using Eqs. (32) and (33), we can evaluate the integral appearing in the first term of Eqs. XXX: 5

$$\int_\tau^t d\tau^*\, p_i^{F*}(\tau^*;\tau,\tau,\bar{u},\vec{\tau}_{\varnothing},\varnothing)p_{ij}^F(t;\tau^*,\tau,\bar{g}_i(\tau^*-\tau,\bar{u}),\tau_F,\{F\})$$
$$= \int_\tau^t f_i^F(\tau^*-\tau;\bar{u})h_{ij}^F(t-\tau^*;\bar{g}_i(\tau^*-\tau,\bar{u}))\, d\tau^* \times \prod_{G\neq F} (1 - F_i^G(t-\tau;\bar{u})) . \quad (41)$$

The interpretation of Eq. (41) is straightforward as it gives the probability per unit time of a transition due to the F-induced event between configurations i and j after a time $t-\tau$ if no stimulus other than F is activated on this time interval.

We can also see that the introduction of Eq. (39) into the last term of Eq. (40) leads again to an integral of the activation density and to two terms: one is again associated with the initial probability density and the other one with the ingoing density, but both correspond to the activation of two stimuli on the time interval considered, one of which induces the event responsible for the change of configuration. The subsequent substitutions of the activation density with Eq. (39) give the development of Eq. (16) with respect to the number of stimuli activated at the time the first delay is elapsed, i.e.,

$$q_{ij}^F(t;\bar{u}) = \int_o^t f_i^F(\tau;\bar{u})h_{ij}^F(t-\tau;\bar{g}_i(\tau,\bar{u}))\,d\tau \prod_{G\neq F}\left[1-\int_o^t dt'\int_o^{t'} d\tau\, f_i^G(\tau;\bar{u})h_i^G(t'-\tau;\bar{g}_i(\tau,\bar{u}))\right]$$
$$= \int_o^t f_i^F(\tau;\bar{u})h_{ij}^F(t-\tau;\bar{g}_i(\tau,\bar{u}))\,d\tau$$
$$\times\left[\prod_{G\neq F}(1-F_i^G(t;\bar{u})) + \sum_{G\neq F}\int_o^t f_i^G(\tau;\bar{u})(1-H_i^G(t-\tau;\bar{g}_i(\tau,\bar{u})))\,d\tau\right.$$
$$\left.\times\prod_{H\neq F,G}(1-F_i^H(t;\bar{u})) + \sum_{G\neq F}\sum_{H\neq F,G}\ldots\right] . \tag{42}$$

Therefore Eq. (40) becomes equivalent to Eq. (22). We have thus shown that by suppressing the possibility for stimuli to remain activated after a change of dynamics, the forward non-Markov treatment of Sec. IV.C reduces to the forward semi-Markov theory given in Sec. III.B.2. Further simplifying the problem, we can now assume that there is no delay in the realization of an event induced by the activation of a stimulus. This means that

$$h_{ij}^F(t;\bar{u}) = \hat{p}_F(i\to j\,|\,\bar{u})\cdot\delta(t) \ . \tag{43}$$

Consequently, Eq. (16) takes the following form:

$$q_{ij}^F(t;\bar{u}) = f_i^F(t;\bar{u})\hat{p}_F(i\to j\,|\,\bar{g}_i(\tau,\bar{u}))$$
$$\times\prod_{G\neq F}[1-F_i^G(t;\bar{u})] \ , \tag{44}$$

which highlights the simple competition between stimulus activations. If the competing processes in Eq. (44) now correspond to the different possible transitions out of dynamics i, we find the classical semi-Markov form of the TPD. The stimulus-driven theoretical extensions are thus consistent with the previous theory.

V. A TEST CASE

V.A. Problem Description

In order to illustrate the previous developments, we consider in this section a basic model for the pressurization of containment, which is caused by an inner combustion whose duration t_H is distributed according to the pdf $f_H(t_H)$. The corresponding (linear) pressure rise can be mitigated by the opening of a relief valve, at a pressure level P_v distributed according to $f_v(P_v)$. The support of this distribution is $[P_v^o, P_v^{max}]$. This valve should avoid a catastrophic rupture of the containment taking place at $P = P_c$, where P_c has a pdf $f_c(P_c)$ on the interval $[P_c^o, P_c^{max}]$. Capital letters will be used as usual for the corresponding cdf's.

The evolution equation for the containment pressure is written

$$\frac{dP}{dt} = c\theta(t_H - t) - \kappa_v P\cdot\theta_v \ , \tag{45}$$

where θ_v is a Boolean variable changing its value from 0 to 1 once level P_v has been crossed. Constants c and κ_v are such that $P_v^o < P_c^o < c/\kappa_v$ by assumption.

The possible dynamics of this system are thus

- $dP/dt = c$ ($i = 1$) as long as the combustion goes on before the valve opening or the rupture
- $dP/dt = o$ ($i = 2$) when the pressurization comes to an end before any other event
- $dP/dt = c - \kappa_v P$ ($i = 3$) when the relief valve is opened before the end of the combustion process
- $dP/dt = -\kappa_v P$ ($i = 4$) when the pressurization is stopped after the valve opening.

For convenience, we also define a fifth absorbing configuration, corresponding to exceeding level P_c and where the dynamics becomes indifferent since it is associated with the containment rupture. Note that the latter situation can be reached only either from configuration 1 or from configuration 3 provided the valve opening takes place at $P_v < c/\kappa_v$. Indeed, with this condition, the pressure keeps increasing in dynamics $i = 3$ according to

$$t - t_v = \frac{1}{\kappa_v}\ln\left(\frac{c-\kappa_v P_v}{c-\kappa_v P(t\,|P_v)}\right) \ , \tag{46}$$

where t_v is such that $P(t_v|P_v) = P_v = ct_v$ in dynamics 1; $P(t|P_v)$ reaches an asymptotic limit $P_\infty = c/\kappa_v$.

V.B. Analytical Estimation of the Probability of Catastrophic Rupture

Following the discussion on the possible accident situations made hereabove, we can separate two cases and therefore write $P_{rupt}(t)$, the probability of containment rupture as a function of time, as the sum of two contributions, corresponding to the system evolving in dynamics 1 and 3 when the rupture occurs, respectively:

$$P_{rupt}(t) = P_1(t) + P_3(t) \ . \tag{47}$$

V.B.1. Rupture Threshold Exceeded in Dynamics 1 at Time t

This situation requires the simultaneous occurrence of three events:

- The combustion time must be larger than t.
- The relief valve is not yet opened at the rupture limit P_c.
- The time elapsed since the beginning of the transient must be large enough for the pressure to have reached P_c in dynamics 1.

Averaging the probability of this intersection of events over the distribution of P_c, we obtain for $P_1(t)$

$$P_1(t) = \int_{P_c^o}^{P_c^{max}} f_c(P_c)\cdot\left(1 - F_H\left(\frac{P_c}{c}\right)\right) \times (1 - F_v(P_c))\cdot\theta\left(t - \frac{P_c}{c}\right) dP_c \tag{48}$$

since the pressure in dynamics 1 is linearly increasing, starting from $P(o) = 0$.

V.B.2. Rupture Threshold Exceeded in Dynamics 3 at Time t

The valve opening, causing the transition in this state from configuration 1, must take place at a pressure level $P_v < c/\kappa_v$, in order to keep a positive pressure derivative in dynamics 3.

The combustion duration must again be larger than the time $t_c(P_c|P_v)$ to reach the rupture pressure P_c, given P_v. From Eq. (46), t_H must satisfy

$$t_H > t_c(P_c|P_v) = t_v + \frac{1}{\kappa_v}\ln\left(\frac{c - \kappa_v P_v}{c - \kappa_v P_c}\right) = \frac{P_v}{c} + \frac{1}{\kappa_v}\ln\left(\frac{c - \kappa_v P_v}{c - \kappa_v P_c}\right) \ . \tag{49}$$

Rupture in dynamics 3 also implies that $P_v < P_c$.

To state more clearly the condition on the minimum time required to attain the undesired event, we split the cases where $P_v \leq P_c^o$ and $P_v \geq P_c^o$. Then,

$$\begin{aligned} P_3(t) = & \int_{P_v^o}^{P_c^o} dP_v \ f_v(P_v)\cdot\theta(P(t|P_v) - P_c^o) \\ & \times \int_{P_c^o}^{P(t|P_v)} dP_c \ f_c(P_c)\cdot(1 - F_H(t_c(P_c|P_v))) \\ & + \int_{P_c^o}^{c/\kappa_v} dP_v \ f_v(P_v)\cdot\theta\left(t - \frac{P_v}{c}\right) \\ & \times \int_{P_v}^{P(t|P_v)} dP_c \ f_c(P_c)\cdot(1 - F_H(t_c(P_c|P_v))) \\ = & \int_{P_v^o}^{P_c^o} dP_v \ f_v(P_v)\cdot\theta(P(t|P_v) - P_c^o) \\ & \times \int_{P_c^o}^{P(t|P_v)} dP_c \ f_c(P_c)\cdot(1 - F_H(t_c(P_c|P_v))) \\ & + \theta\left(t - \frac{P_c^o}{c}\right)\int_{P_c^o}^{min(c/\kappa_v, ct)} dP_v \ f_v(P_v) \\ & \times \int_{P_v}^{P(t|P_v)} dP_c \ f_c(P_c)\cdot(1 - F_H(t_c(P_c|P_v))) \ , \end{aligned} \tag{50}$$

where $P(t|P_v)$ is given by Eq. (46).

V.B.3. Event Tree of the Pressurization Case

The possible evolutions of the system are summarized in the event tree presented in Fig. 3. It should be observed that

- headers are defined on both process variables and uncertain parameters
- the transition between discrete states 1 and 3 can lead to completely different outcomes, depending on the value of the uncertain parameters.

Such characteristics cannot be dealt with using classical analysis techniques. Section V.C displays how the stimulus-based framework is capable of treating this problem.

V.C. Application of the Stimulus-Driven Approach

First, we would like to underline the illustrative purpose of this treatment for the present problem. Indeed, the analytical expressions given in Sec. V.B are quite easily deduced from the analysis of the potential scenarios, while the introduction of the concept of stimulus brings more complexity in the developments below. This section has thus to be considered as a proof of the coherence of the theory, the utility of which fully appears 3

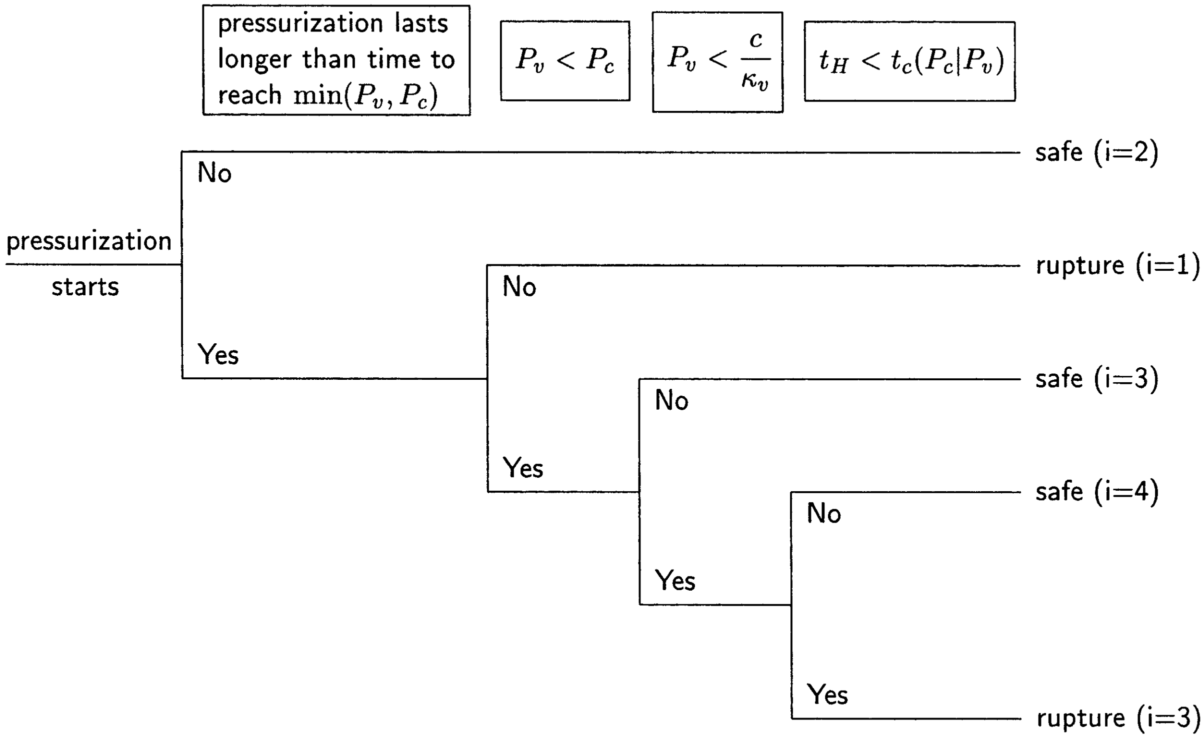

Fig. 3. Event tree of the pressurization test case.

when it becomes the support of an automatic generation of accident sequences and of their probabilistic assessment.

V.C.1. Inventory of Stimuli and Induced Events

Let

- F_1 denote the end of the combustion process, which causes the end of the containment pressurization
- F_2 correspond to the crossing of the minimum relief valve pressure threshold P_v^o, from which the valve opening can be actuated
- F_3 be equivalent to F_2, for the minimum rupture pressure P_c^o, and the actual rupture that can follow.

The distributions for the activation times and time delays are now to be defined in the different dynamics the system can evolve in. One can directly notice that none of the stimuli can be activated in both configurations 2 and 4, what we express by writing

$$f_i^{F_j}(t) = \delta(t - t_{AD}) \ , \quad j = 1,2,3, \; i = 2,4 \ , \qquad (51)$$

where t_{AD} is an upper bound of the accident duration (see Sec. III.B.3). The pdf of the delay is irrelevant in these cases.

In configuration 1, the distribution of the activation time of F_1 is that of t_H, while the pressurization stops instantaneously when F_1 is activated:

$$f_1^{F_1}(t;P') = \frac{f_H\left(t + \dfrac{P'}{c}\right)}{1 - F_H\left(\dfrac{P'}{c}\right)} \quad \text{and} \quad h_1^{F_1}(t;P'') = \delta(t) \qquad (52)$$

if dynamics 1 is entered at P' and stimulus F_1 is activated at P''. The same expressions are also valid in configuration 3.

As for stimuli F_2 and F_3, their activation times are deterministic and defined by the dynamics in the current configuration, while the delay distributions are the direct transposition in the time space of the pdf's of P_v and P_c since the pressure evolution is monotonic: 6

$$f_1^{F_2}(t;P') = \delta\left(t - \frac{P_v^o - P'}{c}\right) \cdot \theta(P_v^o - P')$$

and

$$h_1^{F_2}(t;P'') = cf_v(P'' + ct) \qquad (53)$$

$$f_1^{F_3}(t;P') = \delta\left(t - \frac{P_c^o - P'}{c}\right)\cdot\theta(P_c^o - P')$$

and

$$h_1^{F_3}(t;P'') = cf_c(P'' + ct) \ . \tag{54}$$

In dynamics 3, F_2 cannot be activated [see Eq. (51)], while the deterministic activation of F_3 comes from Eq. (46):

$$f_3^{F_3}(t;P') = \theta(P_c^o - P')\cdot\delta\left(t - \frac{1}{\kappa_v}\ln\left(\frac{c - \kappa_v P'}{c - \kappa_v P_c^o}\right)\right) \ . \tag{55}$$

The distribution of the corresponding delay is again obtained by the change of variable $P \to t$ in $f_c(P_c)$; i.e.,

$$h_3^{F_3}(t;P'') = (c - \kappa_v P'')e^{-\kappa_v t} \times f_c\left(\frac{c}{\kappa_v} - \left(\frac{c}{\kappa_v} - P''\right)e^{-\kappa_v t}\right) \ . \tag{56}$$

Let us remark here that the definition of the stimuli is not unique. We could indeed have referred to a stimulus F_1' corresponding to the start of the pressurization when entering dynamics 1. The delay in this latter case would have been the combustion time. Mathematically, this would simply result in the permutation of the expressions of $f_1^{F_1}$ and $h_1^{F_1}$ in Eq. (52).

V.C.2. Rupture Probability and Ingoing Densities

Having defined in the problem description a fifth configuration that is entered when rupture occurs, we can assimilate the rupture probability to that of being in state 5, no matter what the pressure value is. As this state is absorbing, we can write

$$P_{rupt}(t) = \int_o^t d\tau \int dP\, \varphi(P,5,\tau) \ . \tag{57}$$

Any dependence in a set $\mathcal{A}$ of stimuli remaining activated after entering state 5 is irrelevant here. However, if stimulus F_3 was activated before a transition between configurations 1 and 3, it would remain activated after this change of dynamics. This would require the full non-Markovian treatment. Yet, we have to deal with a delay distribution expressing the time necessary to reach a given zone in the process variables domain. As mentioned in Sec. IV.B, the conditional pdf of the delay takes the form Eq. (31), and a semi-Markovian frame is sufficient. From Eq. (22), we obtain the form of the ingoing density into state 5:

$$\begin{aligned}\varphi(P,5,t) = &\int \pi(P',1,o)\cdot\delta(P - P' - ct)\cdot q_{15}^{F_3}(t;P')\, dP' \\ &+ \sum_{\mathcal{A}} \int_o^t d\tau \int dP'\, \varphi(P',3,\tau,\mathcal{A}) \\ &\times \delta\left(P - \frac{c}{\kappa_v} + \left(\frac{c}{\kappa_v} - P'\right)e^{-\kappa_v(t-\tau)}\right) \\ &\times q_{35}^{F_3}(t-\tau;P',\mathcal{A}) \\ = &\ \delta(P - ct)\cdot q_{15}^{F_3}(t;0) \\ &+ \sum_{\mathcal{A}} \int_o^t d\tau \int dP'\, \varphi(P,3,\tau,\mathcal{A}) \\ &\times \delta\left(P - \frac{c}{\kappa_v} + \left(\frac{c}{\kappa_v} - P'\right)e^{-\kappa_v(t-\tau)}\right) \\ &\times q_{35}^{F_3}(t-\tau;P',\mathcal{A})\end{aligned} \tag{58}$$

where we have accounted for the initial condition $\pi(P',1,o) = \delta(P')$ and where the sum on $\mathcal{A}$ is limited to the empty set and $\{F_3\}$, given the physics of the problem. We also have

$$\begin{aligned}\varphi(P,3,t,\mathcal{A}) = &[\theta(P - P_c^o)\delta_{\mathcal{A},\{F_3\}} + \theta(P_o^c - P)\delta_{\mathcal{A},\varnothing}] \\ &\times \int dP'\, \pi(P',1,o) \\ &\times \delta(P - P' - ct)\cdot q_{13}^{F_2}(t;P') \\ = &[\theta(P - P_c^o)\delta_{\mathcal{A},\{F_3\}} + \theta(P_o^c - P)\delta_{\mathcal{A},\varnothing}] \\ &\times \delta(P - ct)\cdot q_{13}^{F_2}(t;0) \ .\end{aligned} \tag{59}$$

Note that there is no dependence on $\mathcal{A}$ in the expression of the probabilities per unit time of the transitions leaving dynamics 1 because no stimulus can be initially activated. From Eq. (16), we find

$$\begin{aligned}q_{15}^{F_3}(t;0) = &\int_o^t \delta\left(\tau - \frac{P_c^o}{c}\right)\cdot cf_c(P_c^o + c(t-\tau))\, d\tau \\ &\times \left[1 - \int_o^t dt' f_H(t')\right] \\ &\times \left[1 - \int_o^t dt' \int_o^{t'} d\tau\, \delta\left(\tau - \frac{P_v^o}{c}\right) \times cf_v(P_v^o + c(t'-\tau))\right] \\ = &\ c\cdot f_c(ct)\cdot\theta(ct - P_c^o) \\ &\times (1 - F_H(t))\cdot(1 - F_v(ct))\end{aligned} \tag{60}$$

and similarly

$$q_{13}^{F_2}(t;0) = c \cdot f_v(ct) \cdot \theta(ct - P_v^o) \cdot (1 - F_H(t)) \cdot (1 - F_c(ct)) \ . \tag{61}$$

As for $q_{35}^{F_3}$, we can observe that the dependence on $\mathcal{A}$ amounts to using Heaviside stepfunctions on the value of the process variable P, thereby allowing one to replace the sum on $\mathcal{A}$ in Eq. (58) with a unique expression:

$$\begin{aligned} q_{35}^{F_3}(t;P') = &\left\{\theta(P_c^o - P') \int_o^t f_3^{F_3}(\tau;P') \cdot h_3^{F_3}\left(t - \tau; \frac{c}{\kappa_v} - \left(\frac{c}{\kappa_v} - P'\right) \cdot e^{-\kappa_v \tau}\right) d\tau \right. \\ &\left. + \theta(P' - P_c^o) \cdot \int_o^t \delta(\tau) \cdot \frac{h_3^{F_3}\left(\dfrac{P' - P_c^o}{c} + t - \tau; \tilde{P}(P')\right)}{1 - F_c(P')} d\tau \right\} \\ &\times \left[1 - \int_o^t dt' \int_o^{t'} d\tau \, \frac{f_H\left(\tau + \dfrac{P'}{c}\right)}{1 - F_H\left(\dfrac{P'}{c}\right)} \cdot \delta(t' - \tau)\right] , \end{aligned} \tag{62}$$

where Eq. (29) has been used for the conditional delay density in the case $P' > P_c^o$. Indeed, when stimulus F_3 is activated before the transition $1 \to 3$, the virtual initial pressure $\tilde{P}(P')$ (which was introduced in Sec. IV.B) is that obtained when evolving backward in dynamics 3 during a time interval $(P' - P_c^o)/c$, starting from P'; i.e.,

$$c - \kappa_v P' = (c - \kappa_v \tilde{P}(P')) \cdot e^{-\kappa_v \cdot [(P' - P_o^c)/c]} \ . \tag{63}$$

Accounting for Eqs. (55) and (56), we can write explicitly

$$\begin{aligned} q_{35}^{F_3}(t;P') = &\left\{\theta(P_c^o - P')\theta\left(t - \frac{1}{\kappa_v}\ln\left(\frac{c - \kappa_v P'}{c - \kappa_v P_o^c}\right)\right) \cdot (c - \kappa_v P') \cdot e^{-\kappa_v t} \cdot f_c\left(\frac{c}{\kappa_v} - \left(\frac{c}{\kappa_v} - P'\right) \cdot e^{-\kappa_v t}\right) \right. \\ &\left. + \theta(P' - P_c^o) \cdot \frac{(c - \kappa_v \tilde{P}(P')) \cdot e^{-\kappa_v([(P' - P_o^c)/c] + t)} f_c\left(\dfrac{c}{\kappa_v} - \left(\dfrac{c}{\kappa_v} - \tilde{P}(P')\right) \cdot e^{-\kappa_v([(P' - P_o^c)/c] + t)}\right)}{1 - F_c(P')} \right\} \\ &\times \frac{1 - F_H\left(t + \dfrac{P'}{c}\right)}{1 - F_H\left(\dfrac{P'}{c}\right)} \ . \end{aligned} \tag{64}$$

Using Eqs. (58) through (64) and integrating carefully the Dirac peaks, we find the following for the ingoing density in state 5:

$$\begin{aligned} \varphi(P,5,t) = &\ \delta(P - ct) \cdot c f_c(ct) \cdot (1 - F_H(t)) \cdot (1 - F_v(ct)) \cdot \theta(ct - P_c^o) \\ &+ \int_{P_v^o}^{P_c^o} dP' \, \delta\left(P' - ct + \frac{c}{\kappa_v}\ln\left(\frac{c - \kappa_v P'}{c - \kappa_v P}\right)\right) \cdot c f_v(P')\theta(P - P') \cdot f_c(P) \\ &\times (1 - F_H(t)) \cdot \theta(P - P_c^o) \cdot \theta\left(\frac{c}{\kappa_v} - P\right) \\ &+ \int_{P_c^o}^{c/\kappa_v} dP' \, \delta\left(P' - ct + \frac{c}{\kappa_v}\ln\left(\frac{c - \kappa_v P'}{c - \kappa_v P}\right)\right) \cdot c f_v(P')\theta(P - P') \cdot f_c(P) \\ &\times (1 - F_H(t)) \cdot \theta\left(\frac{c}{\kappa_v} - P\right) \ . \end{aligned} \tag{65}$$

One can then easily check that the term-by-term integration of $\varphi(P,5,t)$ according to Eq. (57) gives back the results Eqs. (48) and (50).

VI. CONCLUSIONS

Dynamic approaches to PRA have given considerable insight into the accident sequence delineation of an event tree by modeling more neatly how the competition processes between branching events are driven by the system dynamics in degraded working modes. However, when the occurrence of some events is substantially delayed after their actual triggering, the classical theory of probabilistic dynamics turns out to fall short of the methodological challenge entailed by such a situation.

The theoretical extensions developed in this paper aim at tackling this more complex modeling of competing events. They are based on the concept of stimulus, which needs to be activated before an event can actually occur after some delay. The next event to take place therefore corresponds to the minimum total time necessary for the associated stimulus to be activated and for the delay to be elapsed. This extension of the theory was straightforwardly achieved in a semi-Markov framework, but the latter restriction implies that all stimuli are disactivated after each change of configuration. When this assumption is not satisfied in practice, information on the past history of the system in the transient development must be kept in memory. A further non-Markovian extension had thus to be realized, and both forward and backward cases were considered in establishing the corresponding evolution equations of the process.

These theoretical extensions were shown to be fully compatible with previous, more limited dynamic approaches to PSA, such as the automatic generation of accident sequences based on the crossing of setpoints and used in level-1 integrated PRA. A test case with a fully analytical solution was also developed to display the coherence and capabilities of the new approach.

The numerical challenge induced by this advanced theory of dynamic reliability is of course even larger than the one that was entailed by the classical TPD. Yet, recent years have shown that this obstacle could be overcome with the development of computer technology and of appropriate solution schemes. We believe on this basis that an optimistic forecast can be emitted for this stimulus-driven approach. Anyway, it already gives a theoretical framework from which simplifications can be made and the quality of approximate solution techniques assessed.

APPENDIX A

FACTORIZATION OF THE TRANSITION PROBABILITY PER UNIT TIME

Let $f_i(t;\bar{u})$ be the pdf of the sojourn time in dynamics i if the latter was entered at point $\bar{u}$. We have

$$f_i(t;\bar{u}) = \sum_{j\neq i} q_{ij}(t;\bar{u}) \ , \qquad \text{(A.1)}$$

where $q_{ij}(t;\bar{u})$ is the probability per unit time of a transition between dynamics i and j a time t after entering i at $\bar{u}$ (see Sec. II.B).

Let $f_{ij}(t;\bar{u})$ be the pdf of the transition time $i \to j$ if this transition is considered separately, and let $F_{ij}(t;\bar{u})$ be the corresponding cdf. Then,

$$q_{ij}(t;\bar{u}) = f_{ij}(t;\bar{u}) \cdot \prod_{k\neq j} (1 - F_{ik}(t;\bar{u})) \qquad \text{(A.2)}$$

since the transition $i \to j$ will take place at time t if and only if the other transitions out of i have not yet occurred at this instant.

The transition rate between i and j, given i is entered at point $\bar{u}$, is written

$$p(i \to j, t;\bar{u}) = \frac{f_{ij}(t;\bar{u}) \cdot \prod_{k\neq j} (1 - F_{ik}(t;\bar{u}))}{\prod_{k} (1 - F_{ik}(t;\bar{u}))} = \frac{f_{ij}(t;\bar{u})}{1 - F_{ij}(t;\bar{u})} \ . \qquad \text{(A.3)}$$

Note that these transition rates are explicitly dependent on time. Moreover, the dependence on the process variables is mentioned with respect to the starting point in the system configuration that is left and not as usual with respect to the value of these variables at the transition time. The bridge between both notations is done by the deterministic evolution in i.

From Eqs. (A.1), (A.2), and (A.3), the corresponding transition probability is directly deduced:

$$\hat{p}(i \to j, t;\bar{u}) = \frac{p(i \to j, t;\bar{u})}{\sum_k p(i \to k, t;\bar{u})} = \frac{q_{ij}(t;\bar{u})}{f_i(t;\bar{u})} \ . \qquad \text{(A.4)}$$

The factorization between the distribution of the sojourn time in a configuration and the transition probabilities out of this configuration, which is suggested in Sec. II.B, can indeed lead to

$$f_i(t;\bar{u})\hat{p}(i \to j, \bar{x}) \cdot \delta(\bar{x} - \bar{g}_i(t,\bar{u})) = q_{ij}(t;\bar{u}) \cdot \delta(\bar{x} - \bar{g}_i(t,\bar{u})) \qquad \text{(A.5)}$$

if the ratios $q_{ij}(t;\bar{u})/f_i(t;\bar{u})$ have no explicit time dependence, i.e., if they depend on time only through the dynamic evolution $\bar{g}_i(t;\bar{u})$. In this last expression, we have used again the classical writing of the transition rate, with no time dependence and the condition on the value of the process variables at the transition time. This explicit independence of time is the condition to be satisfied in order to use the outgoing density in the semi-Markov modeling.

APPENDIX B

BACKWARD TREATMENT

B.I. MARKOVIAN BACKWARD TPD

The backward counterpart[3] of the C.K. equation [see Eq. (4)] is written

$$\begin{aligned}\pi(\bar{x}, i, t \mid \bar{x}_o, k, t_o) &= \delta_{ik} e^{-\int_{t_o}^{t} \lambda_i(\bar{g}_i(s - t_o, \bar{x}_o))\, ds} \cdot \delta(\bar{x} - \bar{g}_i(t - t_o, \bar{x}_o)) \\ &\quad + \sum_{j \neq k} \int_{t_o}^{t} p(k \to j \mid \bar{g}_k(\tau - t_o, \bar{x}_o)) \\ &\quad \times e^{-\int_{t_o}^{\tau} \lambda_k(\bar{g}_k(s - t_o, \bar{x}_o))\, ds} \\ &\quad \times \pi(\bar{x}, i, t \mid \bar{g}_k(\tau - t_o, \bar{x}_o), j, \tau)\, d\tau \ , \qquad \text{(B.1)}\end{aligned}$$

where $\pi(\bar{x}, i, t \mid \bar{x}_o, k, t_o)$ stands for the pdf of finding the system in state $(\bar{x}, i)$ at time t, given it was in state $(\bar{x}_o, k)$ at time t_o.

B.II. BACKWARD APPROACH TO THE SEMI-MARKOV MODELING OF DYNAMIC RELIABILITY

Keeping in mind the fact that reference must be made to the transition between two plant configurations, we reinterpret the conditional pdf $\pi(\bar{x}, i, t \mid \bar{x}_o, k, t_o)$ as the probability density of being in state $(\bar{x}, i)$ at time t, given the plant *entered* dynamics k at t_o with process variables $\bar{x}_o$. With this peculiar meaning of the conditional pdf, Eq. (B.1) becomes

$$\begin{aligned}\pi(\bar{x}, i, t \mid \bar{x}_o, k, t_o) &= \delta_{ik}(1 - F_i(t - t_o; \bar{x}_o))\delta(\bar{x} - \bar{g}_i(t - t_o, \bar{x}_o)) \\ &\quad + \sum_{j \neq k} \int_{t_o}^{t} q_{kj}(\tau - t_o; \bar{x}_o) \\ &\quad \times \pi(\bar{x}, i, t \mid \bar{g}_k(\tau - t_o, \bar{x}_o), j, \tau)\, d\tau \ , \qquad \text{(B.2)}\end{aligned}$$

where $F_i(t; \bar{x}_o)$ is the cdf of the sojourn time in configuration i, entered at $\bar{x}_o$, associated with Eq. (A.1).

B.III. SEMI-MARKOV BACKWARD TREATMENT OF STIMULUS ACTIVATIONS AND DELAYS

We must now generalize Eq. (B.2) to the case of stimulus-driven transitions. Using Eqs. (16) and (24), we readily find

$$\begin{aligned}\pi(\bar{x}, i, t \mid \bar{x}_o, k, t_o) &= \delta_{ik}(1 - P_i(t - t_o; \bar{x}_o))\delta(\bar{x} - \bar{g}_i(t - t_o, \bar{x}_o)) \\ &\quad + \sum_{j \neq k} \sum_{F} \int_{t_o}^{t} q_{kj}^{F}(\tau - t_o; \bar{x}_o) \\ &\quad \times \pi(\bar{x}, i, t \mid \bar{g}_k(\tau - t_o, \bar{x}_o), j, \tau)\, d\tau \ , \qquad \text{(B.3)}\end{aligned}$$

where the condition $\bar{x}_o, k, t_o$, which refers to the entry in the plant configuration, relates directly to the definition of $q_{kj}^{F}(t - t_o; \bar{x}_o)$.

B.IV. GENERAL BACKWARD NON-MARKOV TREATMENT

The backward treatment can also be applied in this case if we use the following conditional pdf:

$$\pi(\bar{x}, i, t \mid \bar{u}^*, k, \tau^*, \tau, \vec{\tau}_{\mathcal{A}}, \mathcal{A}) \ , \qquad \text{(B.4)}$$

which is the probability density to find the system in $(\bar{x}, i, t)$, given it underwent an event (change of dynamics or stimulus activation) in state k at $(\bar{u}^*, \tau^*)$, resulting in a set $\mathcal{A}$ of activated stimuli, and given state k was entered at τ. Similarly to what was done in the forward case, an explicit reference to the time τ^* of the last event is done, even if it will always be equal either to the entry time τ or to one of the components of $\vec{\tau}_{\mathcal{A}}$.

The unconditional pdf is directly obtained from

$$\begin{aligned}\pi(\bar{x}, i, t) &= \sum_{k} \int d\bar{u}^* \int_{o}^{t} d\tau \int_{\tau}^{t} d\tau^*\, \pi(\bar{x}, i, t \mid \bar{u}^*, k, \tau^*, \tau, \vec{\tau}_{\varnothing}, \varnothing) \\ &\quad \times \delta(\tau)\delta(\tau - \tau^*)\pi(\bar{u}^*, k, \tau) \ . \qquad \text{(B.5)}\end{aligned}$$

The conditional pdf [Eq. (B.4)] is obtained either from a "free flight" of the system without any event taking place between τ^* and t or from any next event occurring at an intermediate time $s^* \in [\tau^*, t]$ (see Fig. 4), from which the conditional pdf is further considered.

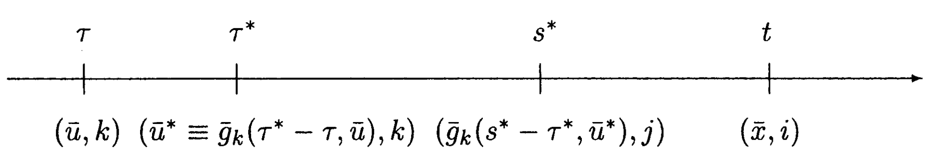

Fig. 4. Evolution of the process variables and configurations up to a next event.

In mathematical terms, we write

$$\begin{aligned}
&\pi(\bar{x}, i, t \,|\, \bar{u}^*, k, \tau^*, \tau, \vec{\tau}_{\mathcal{A}}, \mathcal{A}) \\
&\quad = \delta_{ik} \cdot \delta(\bar{x} - \bar{g}_i(t - \tau^*, \bar{u}^*))(1 - P_i(t; \tau^*, \tau, \bar{u}^*, \vec{\tau}_{\mathcal{A}}, \mathcal{A})) \\
&\qquad + \sum_{G \notin \mathcal{A}} \sum_j \int_{\tau^*}^{t} ds^* \int_{\tau}^{s^*} ds \int_{\tau^*}^{t} d\tau_G \, \pi(\bar{x}, i, t \,|\, \bar{g}_k(s^* - \tau^*, \bar{u}^*), j, s^*, s, \vec{\tau}_{\mathcal{A}+\{G\}}, \mathcal{A} + \{G\}) \\
&\qquad \times \delta_{jk} \delta(s - \tau) \delta(s^* - \tau_G) p_k^{G*}(s^*; \tau^*, \tau, \bar{u}^*, \vec{\tau}_{\mathcal{A}}, \mathcal{A}) \\
&\qquad + \sum_{\mathcal{A}'} \sum_j \int_{\tau^*}^{t} ds^* \int_{\tau}^{s^*} ds \, \pi(\bar{x}, i, t \,|\, \bar{g}_k(s^* - \tau^*, \bar{u}^*), j, s^*, s, \vec{\tau}_{\mathcal{A}'}, \mathcal{A}') \\
&\qquad \times \delta(s - s^*) \sum_{G \in \mathcal{A}} p_{kj}^{G}(s^*; \tau^*, \tau, \bar{u}^*, \vec{\tau}_{\mathcal{A}}, \mathcal{A}) \delta_{kj}^{G}(\mathcal{A} \to \mathcal{A}') \\
&\quad = \delta_{ik} \cdot \delta(\bar{x} - \bar{g}_i(t - \tau^*, \bar{u}^*))(1 - P_i(t; \tau^*, \tau, \bar{u}^*, \vec{\tau}_{\mathcal{A}}, \mathcal{A})) \\
&\qquad + \sum_{G \notin \mathcal{A}} \int_{\tau^*}^{t} d\tau_G \, \pi(\bar{x}, i, t \,|\, \bar{g}_k(\tau_G - \tau^*, \bar{u}^*), k, \tau_G, \tau, \vec{\tau}_{\mathcal{A}+\{G\}}, \mathcal{A} + \{G\}) \cdot p_k^{G*}(\tau_G; \tau^*, \tau, \bar{u}^*, \vec{\tau}_{\mathcal{A}}, \mathcal{A}) \\
&\qquad + \sum_{\mathcal{A}'} \sum_j \int_{\tau^*}^{t} ds \, \pi(\bar{x}, i, t \,|\, \bar{g}_k(s - \tau^*, \bar{u}^*), j, s, s, \vec{\tau}_{\mathcal{A}'}, \mathcal{A}') \sum_{G \in \mathcal{A}} p_{kj}^{G}(s; \tau^*, \tau, \bar{u}^*, \vec{\tau}_{\mathcal{A}}, \mathcal{A}) \delta_{kj}^{G}(\mathcal{A} \to \mathcal{A}') \ . \qquad \text{(B.6)}
\end{aligned}$$

Though one could be reluctant to consider Eq. (B.6) in detail, its interpretation is straightforward. The first term corresponds to the abovementioned free flight, where the system keeps evolving in the same state, while no activated stimulus leads to a change in dynamics and no additional stimulus activation takes place. The integral terms make the bridge between the condition $(\bar{u}^*, k, \tau^*, \tau, \mathcal{A})$ and a condition $(\bar{g}_k(s^* - \tau^*, \bar{u}^*), j, s^*, s, \mathcal{A}')$, posterior in time and corresponding to the status of the system after a first event, either a new activation or a change in dynamics. In the first case, the state and entry time are of course conserved. In the second case, the last event is from now on a change in dynamics before any other event can take place and leads the system into state j, with the stimuli remaining activated forming set $\mathcal{A}'$. This time, there is no need to integrate any component of $\vec{\tau}_{\mathcal{A}}$ since this vector appears as a conditioning variable in the pdf and not as one of its arguments, such as in the forward case.

APPENDIX C

ADAPTATION OF THE MODELING IN THE ASSUMPTION OF FACTORIZATION

C.I. SEMI-MARKOV TREATMENT OF STIMULUS-ACTIVATED TRANSITIONS

Some simplification can be brought to the developments given in Eqs. (22), (23), and (24) if we adopt again the assumption of factorization between the distribution of the sojourn time in a plant configuration i and the transition probabilities out of i (see Appendix A). In this case, we have to consider the probability $p_i^F(t; \bar{u})\, dt$ of leaving dynamics i because of the occurrence of the event associated with F, after a time dt about t, given this configuration was entered at $\bar{u}$. Indexing the transition probabilities with the stimulus causing the transition, Eq. (22) becomes

$$\varphi(\bar{x}, i, t) = \sum_F \sum_{j \neq i} \int_o^t d\tau \int d\bar{u} \, [\pi(\bar{u}, j, \tau)\delta(\tau) + \varphi(\bar{u}, j, \tau)] \delta(\bar{x} - \bar{g}_j(t - \tau, \bar{u})) p_j^F(t - \tau; \bar{u}) \hat{p}_F(j \to i \,|\, \bar{x}) \ . \qquad \text{(C.1)}$$

Alternatively, we can introduce, within this assumption of factorization, the outgoing density $\psi_F(\bar{x}, i, t)$, conditional to stimulus F having triggered the event causing the branching out of i. This outgoing density obeys an adapted form of Eq. (4):

$$\begin{aligned}
\psi_F(\bar{x}, i, t) = \sum_{j \neq i} \sum_G \int_o^t d\tau \int d\bar{u} \left[\pi(\bar{u}, i, \tau)\delta(\tau) + \sum_G \psi_G(\bar{u}, j, \tau) \cdot \hat{p}_G(j \to i \,|\, \bar{u}) \right] \\
\times \delta(\bar{x} - \bar{g}_i(t - \tau, \bar{u})) \cdot p_i^F(t - \tau; \bar{u}) \ . \qquad \text{(C.2)}
\end{aligned}$$

The probability density π can then be expressed in a way equivalent to Eq. (23):

$$\pi(\bar{x},i,t) = \sum_{j\neq i}\int_o^t d\tau \int d\bar{u}\left[\pi(\bar{u},i,\tau)\delta(\tau) + \sum_G \psi_G(\bar{u},j,\tau)\cdot\hat{p}_G(j\to i|\bar{u})\right] \times \delta(\bar{x}-\bar{g}_i(t-\tau,\bar{u}))[1-P_i(t-\tau;\bar{u})] \ .$$

As for the backward formulation Eq. (B.3), in this assumption it becomes

$$\pi(\bar{x},i,t|\bar{x}_o,k,t_o) = \delta_{ik}\delta(\bar{x}-\bar{g}_i(t-t_o,\bar{x}_o))[1-P_i(t-t_o;\bar{x}_o)] + \sum_{j\neq k}\sum_F \int_{t_o}^t p_k^F(\tau-t_o;\bar{x}_o)\hat{p}_F(k\to j|\bar{g}_k(\tau-t_o,\bar{x}_o))\pi(\bar{x},i,t|\bar{g}_k(\tau-t_o,\bar{x}_o),j,\tau)\,d\tau \ . \quad \text{(C.3)}$$

C.II. GENERAL NON-MARKOV TREATMENT

In this case, we can resort again to outgoing densities, which are defined in the following way:

- $\psi_{out}^F(\bar{x},i,t,\vec{\tau}_{\mathcal{A}},\mathcal{A})$ is the outgoing density out of dynamics i at point $\bar{x}$ and time t, via the F-induced event if the state transition is *entered* with a set $\mathcal{A}$ of activated stimuli ($\to F\in\mathcal{A}$)
- $\psi_F(\bar{x},i,t,\tau,\vec{\tau}_{\mathcal{A}},\mathcal{A})$ is the activation density of stimulus F in configuration i, at $(\bar{x},t)$, for an entry in i at τ, and a set $\mathcal{A}$ of stimuli already activated *before* ($\to F\notin\mathcal{A}$). Therefore, the reference to the activation time t is not redundant in this case as it was when modeling the problem with ingoing densities (see Sec. IV.C). Yet, the equality $t=\tau_F$ must be accounted for as soon as set $\mathcal{A}$ is updated.

We also introduce in the transition probabilities the rules of disactivation of the stimuli associated with a given transition, in the form $\hat{p}_G(j\to i|\bar{u};\mathcal{A}'\to\mathcal{A})$, with the subsequent update of $\vec{\tau}_{\mathcal{A}'}$ in $\vec{\tau}_{\mathcal{A}}$. Note that the probability per unit time of a transition out of i via the event induced by stimulus F [see Eq. (33)] no longer depends on the configuration reached after the transition. In order to avoid any confusion with Eq. (32), we will write this quantity $p_{i*}^F(t;\tau^*,\tau,\bar{u}^*,\vec{\tau}_{\mathcal{A}},\mathcal{A})$.

With these notations, if $F\in\mathcal{A}$, we have

$$\begin{aligned}\psi_{out}^F(\bar{x},i,t,\vec{\tau}_{\mathcal{A}},\mathcal{A}) &= \int d\bar{u}\int_o^t d\tau^*\int_o^{\tau^*} d\tau\,\delta(\bar{x}-\bar{g}_i(t-\tau^*,\bar{u}))p_{i*}^F(t;\tau^*,\tau,\bar{u},\vec{\tau}_{\mathcal{A}},\mathcal{A})\\ &\times\Bigg[\sum_{G\in\mathcal{A}}\psi_G(\bar{u},i,\tau^*,\tau,\vec{\tau}_{\mathcal{A}/\{G\}},\mathcal{A}/\{G\})\delta(\tau^*-\tau_G)\\ &+\sum_{\mathcal{A}'\supset\mathcal{A}}\sum_{G\in\mathcal{A}'}\sum_{j\neq i}\int_o^{\tau^*}\ldots\int_o^{\tau^*} d\vec{\tau}_{\mathcal{A}'/\mathcal{A}}\,\psi_{out}^G(\bar{u},j,\tau^*,\vec{\tau}_{\mathcal{A}'},\mathcal{A}')\delta(\tau^*-\tau)\hat{p}_G(j\to i|\bar{u};\mathcal{A}'\to\mathcal{A})\Bigg] \ . \quad \text{(C.4)}\end{aligned}$$

Again, a multiple integral on the components of $\vec{\tau}_{\mathcal{A}'/\mathcal{A}}$, since the corresponding activation times do not condition the future evolution of the system after the change of dynamics. Contributions corresponding to all acceptable activation 8
times for this subset of stimuli must then be accounted for.

From the definition of the outgoing activation density hereabove, we must have this time $F\notin\mathcal{A}$. Then,

$$\begin{aligned}\psi_F(\bar{x},i,t,\tau,\vec{\tau}_{\mathcal{A}},\mathcal{A}) &= \delta_{\mathcal{A},\varnothing}\delta(\tau)\int d\bar{u}\,\pi(\bar{u},i,o)\delta(\bar{x}-\bar{g}_i(t,\bar{u}))p_i^{F*}(t;o,o,\bar{u},\vec{\tau}_{\varnothing},\varnothing)\\ &+\int d\bar{u}\int_\tau^t d\tau^*\,\delta(\bar{x}-\bar{g}_i(t-\tau^*,\bar{u}))p_i^{F*}(t;\tau^*,\tau,\bar{u},\vec{\tau}_{\mathcal{A}},\mathcal{A})\\ &\times\Bigg[\sum_{G\in\mathcal{A}}\psi_G(\bar{u},i,\tau^*,\tau,\vec{\tau}_{\mathcal{A}/\{G\}},\mathcal{A}/\{G\})\delta(\tau^*-\tau_G)\\ &+\sum_{\mathcal{A}'\supset\mathcal{A}}\sum_{G\in\mathcal{A}'}\sum_{j\neq i}\int_o^{\tau^*}\ldots\int_o^{\tau^*} d\vec{\tau}_{\mathcal{A}'/\mathcal{A}}\,\psi_{out}^G(\bar{u},j,\tau^*,\vec{\tau}_{\mathcal{A}'},\mathcal{A}')\delta(\tau^*-\tau)\hat{p}_G(j\to i|\bar{u};\mathcal{A}'\to\mathcal{A})\Bigg] \ . \quad \text{(C.5)}\end{aligned}$$

Again, such as in Eq. (35), we can observe an additional term associated with the first activation in the transient history.

Alternatively to Eq. (36), the probability distribution in configuration i can finally be expressed as

$$\pi(\bar{x},i,t;\mathcal{A}) = \delta_{\mathcal{A},\varnothing}\int d\bar{u}\,\pi(\bar{u},i,o)\delta(\bar{x}-\bar{g}_i(t,\bar{u}))(1-P_i(t;o,o,\bar{u},\vec{\tau}_{\varnothing},\varnothing))$$

$$\times \int d\bar{u}\int_o^t d\tau^*\int_o^{\tau^*} d\tau\int_o^{\tau^*}\dots\int_o^{\tau^*} d\vec{\tau}_{\mathcal{A}}\,\delta(\bar{x}-\bar{g}_i(t-\tau^*,\bar{u}))(1-P_i(t;\tau^*,\tau,\bar{u},\vec{\tau}_{\mathcal{A}},\mathcal{A}))$$

$$\times\Bigg[\sum_{G\in\mathcal{A}}\psi_G(\bar{u},i,\tau^*,\tau,\vec{\tau}_{\mathcal{A}/\{G\}},\mathcal{A}/\{G\})\delta(\tau^*-\tau_G)$$

$$+\sum_{\mathcal{A}'\supset\mathcal{A}}\sum_{G\in\mathcal{A}'}\sum_{j\neq i}\int_o^{\tau^*}\dots\int_o^{\tau^*} d\vec{\tau}_{\mathcal{A}'/\mathcal{A}}\,\psi_{out}^{G}(\bar{u},j,\tau^*,\vec{\tau}_{\mathcal{A}'},\mathcal{A}')\delta(\tau^*-\tau)\hat{p}_G(j\to i|\bar{u};\mathcal{A}'\to\mathcal{A})\Bigg] \;. \quad \text{(C.6)}$$

Finally, we will give the backward form of Eqs. XXX with these assumptions, starting from Eq. (B.6): 5

$$\pi(\bar{x},i,t|\bar{u}^*,k,\tau^*,\tau,\vec{\tau}_{\mathcal{A}},\mathcal{A}) = \delta_{ik}\cdot\delta(\bar{x}-\bar{g}_i(t-\tau^*,\bar{u}^*))(1-P_i(t;\tau^*,\tau,\bar{u}^*,\vec{\tau}_{\mathcal{A}},\mathcal{A}))$$

$$+\int_{\tau^*}^{t} d\tau_G\sum_{G\notin\mathcal{A}} p_k^{G*}(\tau_G;\tau^*,\tau,\bar{u}^*,\vec{\tau}_{\mathcal{A}},\mathcal{A})\pi(\bar{x},i,t|\bar{u}^*,k,\tau_G,\tau,\vec{\tau}_{\mathcal{A}+\{G\}},\mathcal{A}+\{G\})$$

$$+\sum_{j\neq k}\int_{\tau^*}^{t} ds\sum_{G\in\mathcal{A}} p_{k*}^{G}(s;\tau^*,\tau,\bar{u}^*,\vec{\tau}_{\mathcal{A}},\mathcal{A})\sum_{\mathcal{A}'}\hat{p}_G(k\to j|\bar{g}_k(s-\tau^*,\bar{u}^*);\mathcal{A}\to\mathcal{A}')$$

$$\times\,\pi(\bar{x},i,t|\bar{g}_k(s-\tau^*;\bar{u}^*),j,s,s,\vec{\tau}_{\mathcal{A}'},\mathcal{A}') \;, \quad \text{(C.7)}$$

where the assumption of factorization between the distribution of the transition time and the transition probabilities was introduced. As before, we have $\bar{u}^* = \bar{g}_k(\tau^*-\tau,\bar{u})$.

In this expression, we have willingly used in the two integrals two different dummy variables (s^* and s, respectively) in order to highlight the interpretation of each term. The first one corresponds to the activation of a new stimulus, with this last event taking place at time s^*. The second one is associated with a change in dynamics, with s being the entry time in the new state. Both events have to occur after τ^*, where the system is known to be at point $\bar{u}^*$.

ACKNOWLEDGMENTS

The authors dedicate the present extension of the theory of probabilistic dynamics to the memory of Professor Jacques Devooght, whose intellectual influence is perceptible throughout the paper.

They are also indebted to E. Melendez for his valuable input via numerous discussions.

REFERENCES

1. "Severe Accident Risks: An Assessment for Five U.S. Nuclear Power Plants," NUREG-1150, vol. 182, U.S. Nuclear Regulatory Commission (1989).

2. N. SIU, "Risk Assessment for Dynamic Systems: An Overview," *Rel. Eng. Syst. Safety*, **43**, 43 (1994).

3. J. DEVOOGHT and C. SMIDTS, "Probabilistic Reactor Dynamics—I. The Theory of Continuous Event Trees," *Nucl. Sci. Eng.*, **111**, 229 (1992).

4. J. M. IZQUIERDO, E. MELENDEZ, and J. DEVOOGHT, "Relationship Between Probabilistic Dynamics and Event Trees," *Rel. Eng. Syst. Safety*, **52**, 197 (1996).

5. J. DEVOOGHT and C. SMIDTS, "Probabilistic Dynamics as a Tool for Dynamic PSA," *Rel. Eng. Syst. Safety*, **52**, 185 (1996).

6. J. M. IZQUIERDO and M. SANCHEZ-PEREA, "Application of the Integrated Safety Assessment Methodology to the Emergency Procedures of a SGTR of a PWR," *Rel. Eng. Syst. Safety*, **45**, 159 (1994).

7. J. M. IZQUIERDO et al., "An Integrated Approach to Independent Regulatory Evaluations of Nuclear Safety Assessments of Spanish Nuclear Power Stations. Towards Convergence of Technical Nuclear Safety Practices in Europe," presented at EUROSAFE Conf., Paris, France, 2003.

8. P. E. LABEAU, C. SMIDTS, and S. SWAMINATHAN, "Dynamic Reliability: Towards an Integrated Platform for Probabilistic Risk Assessment," *Rel. Eng. Syst. Safety*, **68**, 219 (2000).

9. S. SWAMINATHAN et al., "The Cassini Mission Probabilistic Risk Analysis: Comparison of Two Probabilistic Dynamic Methodologies," *Rel. Eng. Syst. Safety*, **58**, 1 (1997).

10. J. PESCHKE, E. HOFER, M. KLOOS, B. KRZYKACZ-HAUSMANN, and M. SONNENKALB, "Probabilistic Dynamics in Level 2 PSA: An Application of the Stochastic Module MCDET with the Deterministic Dynamic Code MELCOR," Personal Communication (2002).

11. P. E. LABEAU and J. M. IZQUIERDO, "Modeling PSA Problems—II: A Cell-to-Cell Transport Theory Approach," *Nucl. Sci. Eng.*, **150**, ■■■ (2005).

12. J. DEVOOGHT, "Dynamic Reliability," *Adv. Nucl. Sci. Technol.*, 215 (1997).

13. J. DEVOOGHT and C. SMIDTS, "Probabilistic Reactor Dynamics—III. A Framework for Time-Dependent Interaction Between Operator and Reactor During a Transient Involving Human Error," *Nucl. Sci. Eng.*, **112**, 101 (1992).

14. C. SMIDTS, "Probabilistic Reactor Dynamics—IV. An Example of Man/Machine Interaction," *Nucl. Sci. Eng.*, **112**, 114 (1992).

15. D. C. IRVING, "The Adjoint Boltzmann Equation and Its Simulation by Monte Carlo," *Nucl. Eng. Des.*, **15**, 273 (1971).

16. R. M. SUMMERS et al., "Melcor Computer Code Manuals," NUREG/CR-6119 SAND 93-2185, Sandia National Laboratories (1995).

17. F. T. HARPER et al., "Evaluation of Severe Accident Risks: Quantification of Major Input Parameters," NUREG/CR-4551 SAND 86-1309, Vol. 2, Rev. 1, Part 2, Sandia National Laboratories (1991).

18. G. BECKER, J. NAGEL, L. CAMARINOPOULOS, and D. KABRANIS, "Dynamic Reliability Expressed in Terms of Transition Frequency Densities," *Proc. Int. Conf. European Safety and Reliability: Towards a Safer World (ESREL 2001)*, Torino, Italy, September 16–20, 2001, p. 1383, European Safety and Reliability Association (2001).

19. P. E. LABEAU, "Monte Carlo Simulation for Dynamic Reliability Problems with Distributed Safety Borders," *Proc. Int. Conf. European Safety and Reliability: Towards a Safer World (ESREL 2001)*, Torino, Italy, September 16–20, 2001, p. 1395, European Safety and Reliability Association (2001).

20. A. DUBI, "Analytic Approach and Monte Carlo Methods for Realistic Systems Analysis," *Math. Comp. Simul.*, **47**, *2–5*, 243 (1998).

21. A. DUBI, *Monte Carlo Applications in System Engineering*, Wiley, Chichester (2000).

22. P. E. LABEAU and E. ZIO, "The Mathematics of Component-Based and System-Based Monte Carlo Sampling in Realistic System Engineering," *Proc. 2nd Int. Conf. Mathematical Methods in Reliability, Methodology, Practice, and Inference (MMR 2000)*, Bordeaux, France, July 4–7, 2000, p. 667 (2000).